ON BOUNDARY CONDITIONS FOR MULTIDIMENSIONAL HYPERBOLIC SYSTEMS OF CONSERVATION LAWS IN THE FINITE VOLUME FRAMEWORK

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Mai 2003

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Abstract

This report presents a method for imposing boundary conditions in the context of hyperbolic systems of conservation laws in the finite volume framework. This method is particularly well suited for approximations in the framework of Finite Volume Methods in the sense that *it computes directly the normal flux at the boundary* with using just the hyperbolic nature of the system and nothing else. We discuss both linear and non linear problems. In the first group, we consider the wave equation, the Maxwell system and the linear elasticity problem. In the second group, we firstly study conservative systems as the magneto-hydrodynamic system, the Euler equations together with its classical reduced versions : isentropic, isothermal and shallow-water approximations and finally we consider complex (non conservative) models arising in the numerical computation of two fluid models. These latter systems initially motivated our approach.

For each application, we analyze the hyperbolicity and write the eigensystem, then we present the so-called VFFC finite volume approach and provide at the discrete level a general theory for the boundary condition treatment. We analytically and numerically compare our treatment with the incomplete Riemann invariant technique. Finally we address practical issues and we present some numerical results. A last section is devoted to the widely studied one dimensional Euler equations for inviscid flow.

Contents

Notations

- $\mathbb{1}$: Identity matrix
- δ : Kronecker symbol
- $\wedge$: Cross product
- $\otimes$: Tensor product : $(a \otimes b)_{ij} = a_i b_j$
- Δ : Laplacian operator
- $F(v) \cdot \omega$: Normal flux : $F(v) \cdot \omega = \sum_{j=1}^{nd} \omega_i F^i(v)$

1 Introduction

A lot of physical models occurring in continuous mechanics appear to be systems of conservations laws. These equations express in general fundamental laws of physics, namely the conservation of mass, charge, momentum, total energy, etc. When dealing with multidimensional models, essentially the only way of obtaining quantitative results is to use numerical simulation. That is working with a discrete approximation of the system of equations. If one insists on the fact that the conservation laws must be rigorously fulfilled (and according to physical considerations this is often a minimal requirement) one is naturally led to use the so-called finite volume approach. In this paper our aim is to discuss with great details the matter of imposing at the discrete level boundary conditions on the boundary of the computational domain. We shall however always deal first at the continuous level the boundary conditions and then study their discrete version. Note that these boundaries could be either physical (wall, inlet or outlet, etc) or numerical (sometimes called artificial) when *e.g.* the physical domain is infinite. Let us mention that the applications we have in mind are those steaming from Fluid Mechanics, although other fields will also be considered (see Section 5).

The modern theory of partial differential equations relies on Distribution theory and boundary conditions problems have been first studied in this context. A reference in this direction is the famous treatise written in the late sixties by J.L. Lions and E. Magenes [38], [39]. The objective of their program was to study in a systematic way the boundary values problems associated with *linear* partial differential equations. In the same spirit, the book of H.O. Kreiss [36] should be quoted.

Let us deal more particularly with hyperbolic type equations. For these equations, wave propagation phenomena are determinant and we refer to Whitham [60] in which a deep exposition is provided. Concerning the discrete problem associated with these equations, roughly speaking there are two caricatured situations. In the first one, one deals with linear and quiet general (wave) equations while in the second one, one deals with nonlinear particular ones. For linear (wave) equations like *e.g.* the wave equation in 1, 2 or 3 space dimension, Maxwell's equations, Elasticity equations, a lot of works rely on the theoretic and explicit solution to the continuous equations and this leads in general to a non local boundary condition. Then usually, an approximation procedure, *e.g.* an asymptotic expansion with respect to a small parameter, is applied in order to derive local boundary conditions. Next these conditions are discretized by classical difference techniques. This has led to a very wide body of knowledge and we refer to the review article by Tsynkov [55] for numerous results and references. On the opposite, concerning nonlinear equations, most of the works are devoted to particular equations and since no explicit solutions are available, most of the works rely on *ad hoc* procedures usually related to physical considerations.

In this report our aim is to provide a *general theory* for the boundary condition problem at the discrete level for multidimensional hyperbolic systems in the finite volume framework. Our work transfers in a certain sense knowledge in the linear case to the non linear one and gives a systematic approach that can be used in either cases (linear and non linear). Moreover, and in contrast with most of the works on the subject, we do not rely on either specific or computationally expansive functions like *e.g.* Riemann solvers or Riemann invariants. Actually we just use the hyperbolic nature of the system and nothing else. Last but not least, we address directly the key problem which consists in determining directly the *normal flux* on the boundary while other methods rely on the finding of a state on this boundary (a problem that can be ill-posed). As a result of our method, we are able to take into account any physically reasonable boundary condition and Theorem 1 gives a sufficient condition under which the problem of finding the normal flux is well-posed. Moreover this condition is natural and straightforward to check in each example. Although no general result is obtained in the non linear case[1], we are able to treat the case of characteristic boundaries in the context of a wall in fluid mechanics, see Section 6.1. In contrast with methods which require the determination of a state on the boundary, we do not impose artificial conditions since in this case the problem is under determined. Indeed dealing directly with the normal flux leads us to a system with the same number of equations and unknowns, see Section 3.2.3. We shall compare our method with classical ones both from the point of view of computational cost and precision.

[1] Recall that already at the continuous level, this is a well known open problem

2 Setting of the problem in the multidimensional conservative case

Let us consider a system of m conservation equations defined on a nd-dimensional domain Ω (with $nd = 1, 2$ or 3 in practice) :

$$\frac{\partial v}{\partial t} + \nabla \cdot F(v) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+ , \quad v = (v_1, \ldots, v_m) \in \mathbb{R}^m . \tag{2.1}$$

Here F^j maps G into $\mathbb{R}^m$ where G is an open subset of $\mathbb{R}^m$ corresponding to the physically admissible states and

$$\nabla \cdot F(v) = \sum_{j=1}^{nd} \frac{\partial F^j(v)}{\partial x_j} .$$

2.1 Examples of multidimensional conservative problems

Let us give some typical examples (of practical importance) of such systems.

2.1.1 Example 1 : the wave equation

The first one is the multidimensional wave equation that reads :

$$\frac{\partial^2 u}{\partial t^2} - c^2 \triangle u = 0 , \tag{2.2}$$

where c is a given positive real number. Setting $v = (v_0, \ldots, v_{nd})$, $m = nd + 1$, $F_0^j(v) = cv_j$ for $j = 1, \ldots, nd$ and $F_p^j(v) = c\delta_{j,p} v_0$ for $j, p = 1, \ldots, nd$, one sees immediately that system (2.1) implies that $u \equiv v_0$ satisfies (2.2). Hence the system reads here as

$$\frac{\partial u}{\partial t} + c \sum_{\ell=1}^{nd} \frac{\partial v_\ell}{\partial x_\ell} = 0 , \tag{2.3}$$

$$\frac{\partial v_j}{\partial t} + c \frac{\partial u}{\partial x_j} = 0, \quad j = 1, \ldots, nd . \tag{2.4}$$

Observe that

$$\sum_{i=1}^{nd} \omega_i F^i(v) = c(v_\sharp \cdot \omega, u\omega), \quad \forall \omega \in \mathbb{R}^{nd} , \tag{2.5}$$

where we have split $v \equiv (u, v_\sharp)$ with $v_\sharp = (v_1, \ldots, v_{nd})$.

2.1.2 Example 2 : the Maxwell system

This example is Maxwell's system of equations that reads for $nd = 2$ or $nd = 3$:

$$\frac{\partial D}{\partial t} - curl H = 0 , \tag{2.6}$$

$$\frac{\partial B}{\partial t} + curl E = 0 , \tag{2.7}$$

with constitutive equations $D = \epsilon E$ and $B = \mu H$. The vectors D and B live in $\mathbb{R}^{nd}$ and are divergence free vector fields. For simplicity, ϵ and μ are taken as positive constants. Here we take $v = (D, B)$, $m = 2\,nd$ and set

$$\sum_{i=1}^{nd} \omega_i F^i(v) = (-\frac{\omega}{\mu} \wedge B, \frac{\omega}{\epsilon} \wedge D), \quad \forall \omega \in \mathbb{R}^{nd} , \tag{2.8}$$

so that (2.6)-(2.7) is again of the form (2.1).

2.1.3 Example 3 : the linear elasticity problem

The third example is the time dependent linear elasticity problem for $nd = 2$ or $nd = 3$ where motion of the solid is described by the displacement vector $u = (u^1, \cdots, u^{nd})$ as a function of time t and of position $x = (x^1, \cdots, x^{nd})$. Let us remark that the case $nd = 1$ corresponds to the wave equation (2.2). The equations of motion have the form (see for instance Marsden and Hughes [40] and John [35]) :

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{\lambda+\mu}{\rho}\sum_{k=1}^{nd}\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \frac{\mu}{\rho}\sum_{k=1}^{nd}\frac{\partial^2 u_i}{\partial x_k^2}\,, \quad i = 1, \cdots, nd\,, \tag{2.9}$$

with Lamé constants λ, μ and density constant ρ.
Let denote $v_0 = (v_0^1, \cdots, v_0^{nd}) \in \mathbb{R}^{nd}$ and $v_j = (v_j^1, \cdots, v_j^{nd}) \in \mathbb{R}^{nd}$ for $j = 1, \cdots, nd$. Setting $v = (v_0, v_1, \cdots, v_{nd})$, this system is of the form (2.1) with $v_0 \equiv u$ and $m = nd(nd+1)$ as

$$\left\{\begin{array}{rcl} \dfrac{\partial v_0^i}{\partial t} & = & \dfrac{\lambda}{\rho}\displaystyle\sum_{k=1}^{nd}\frac{\partial v_k^k}{\partial x_i} + \frac{\mu}{\rho}\sum_{k=1}^{nd}(\frac{\partial v_k^i}{\partial x_k} + \frac{\partial v_i^k}{\partial x_k}), \quad i = 1, \cdots, nd\,, \\ \dfrac{\partial v_j^i}{\partial t} & = & \dfrac{\partial v_0^i}{\partial x_j}\,, \quad i, j = 1, \cdots, nd\,, \end{array}\right. \tag{2.10}$$

where F^i mapping G into $\mathbb{R}^{nd}$ satisfy

$$\sum_{i=1}^{nd}\omega_i F^i(v) = (\sigma(v_\sharp)\omega, \omega \otimes v_0) \tag{2.11}$$

where we have split $v \equiv (u, v_\sharp)$ with $v_\sharp = (v_1, \ldots, v_{nd})$ and where $\sigma(v_\sharp)$ is the matrix

$$\sigma(v_\sharp) = \frac{\mu}{\rho}\epsilon(v_\sharp) + \frac{\lambda}{\rho}\mathrm{tr}(v_\sharp)\mathbb{1}$$

with $\epsilon(v_\sharp) = \frac{1}{2}({}^t v_\sharp + v_\sharp)$.

2.1.4 Example 4 : the magneto-hydrodynamic system

The fourth application concerns a model of ideal magneto-hydrodynamics. The system satisfied by magnetics fluids reads in conservative form as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \tag{2.12}$$

$$\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \otimes u + P) = 0, \tag{2.13}$$

$$\frac{\partial (\rho E)}{\partial t} + \nabla \cdot (\rho E u + P u) = 0, \tag{2.14}$$

$$\frac{\partial B}{\partial t} + \nabla \cdot (u \otimes B - B \otimes u) = 0\,, \tag{2.15}$$

where the scalar ρ is the density, the vector u in $\mathbb{R}^{nd}$ represents the velocity, the divergence free vector field B (*i.e.* $\nabla \cdot B = 0$) is the magnetic field, the scalar $E = e + \frac{1}{2}|u|^2 + \frac{1}{2\rho}|B|^2$ is the total energy and the matrix P satisfies $P = (p + \frac{1}{2}|B|^2)\mathbb{1} - B \otimes B$. In order to close this system, one should provide an equation of state (EOS) linking the pressure p, the internal energy e and the density ρ. Let us emphasize that this system is of the form (2.1) with $v \in \mathbb{R}^{2nd+2}$, $v = (\rho, \rho u_1, \ldots, \rho u_{nd}, \rho E, B_1, \ldots, B_{nd})$, $m = 2nd + 2$ and for all ω in $\mathbb{R}^{nd}$

$$\sum_{i=1}^{nd}\omega_i F^i(v) = u \cdot \omega(\rho, \rho u, \rho E, -B) + (p + \frac{1}{2}|B|^2)(0, \omega, u \cdot \omega, 0) - B \cdot \omega(0, B, B \cdot u, -u)\,. \tag{2.16}$$

2.1.5 Example 5 : the Euler equations

The third one is the multidimensional Euler equations for compressible inviscid fluids that reads :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \tag{2.17}$$

$$\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \otimes u + p\mathbb{1}) = 0, \tag{2.18}$$

$$\frac{\partial (\rho E)}{\partial t} + \nabla \cdot (\rho E u + p u) = 0, \tag{2.19}$$

where $E = e + \frac{1}{2}|u|^2$ is the total energy and u is a vector in $\mathbb{R}^{nd}$. In order to close this system, one should provide an equation of state (EOS) linking the pressure p, the internal energy e and the density ρ. Let us emphasize that this system reduces to the 3 first equations (2.12)-(2.14) of the magneto-hydrodynamic system where the matrix P is spherical: $P = p\mathbb{1}$, with p denoting the thermodynamical pressure.
This system is of the form (2.1) with $v \in \mathbb{R}^{nd+2}$, $v = (\rho, \rho u_1, \cdots, \rho u_{nd}, \rho E)$, $m = nd + 2$ and

$$\sum_{i=1}^{nd} \omega_i F^i(v) = (u \cdot \omega)v + p(0, \omega, u \cdot \omega), \quad \forall \omega \in \mathbb{R}^{nd}. \tag{2.20}$$

2.1.6 Example 6 : the isentropic compressible Euler system

Introducing the specific entropy function s and the temperature T that satisfy $Tds = de - \frac{p}{\rho^2}d\rho$, it is classical to derive for continuous solutions of (2.17)-(2.18) the fact that the specific entropy is convected by the flow:

$$\frac{\partial s}{\partial t} + u \cdot \nabla s = 0\,.$$

Since $s =$ constant is an obvious solution to this equation, in certain physical cases, it might be relevant to address the reduced system, known as the isentropic compressible Euler system of equations, which consists in the two conservation laws (2.17) and (2.18). This time the EOS is a relation between the pressure and the density: $p = p(\rho)$. This system is of the form (2.1) with $v \in \mathbb{R}^{nd+1}$, $v = (\rho, \rho u_1, \ldots, \rho u_{nd})$, $m = nd + 1$ and

$$F(v) \cdot \omega = u \cdot \omega(\rho, \rho u) + p(0, \omega), \quad \forall \omega \in \mathbb{R}^{nd}. \tag{2.21}$$

The function $p = p(\rho)$ is arbitrary and depends on the fluid which is considered. The only constraint is that the derivative of p with respect to ρ is non negative which corresponds to the physical property that pressure perturbations propagate at finite speed: the speed of sound, $c = \sqrt{\frac{dp}{d\rho}}$. There are two asymptotic cases which are sometimes considered and yield the two following applications.

2.1.7 Example 7 : the isothermal flow

Here the relation between pressure and temperature is linear which physically corresponds to the limit case where the two heat capacities C_v and C_p are equal. In such a case a change in the temperature requires an infinite amount of energy, that is an impossible fact. Hence the temperature cannot vary. Since now the speed of sound is constant, the system (2.17)-(2.18) with the normal flux (2.21) is closed by the EOS: $p = c_0^2\rho$.

2.1.8 Example 8 : the shallow water equations

In that case a very thin layer of fluid is considered. This fluid is in general a liquid and is therefore slightly compressible. In the limit where the fluid is incompressible, the density is constant but the height, h, of the fluid might depends on space and time. In this situation one arrives again to the model (2.17)-(2.18) with the normal flux (2.21) but this time $p = \frac{1}{2}\bar{\rho}gh^2$ and $\rho = \bar{\rho}h$ where $\bar{\rho}$ is the constant density of the fluid and g denotes the gravity.
Of course there are many other examples of systems of conservation laws. We have chosen to study these typical examples because of their physical importance and also because they belong to three different categories. The first one is a scalar second order (with respect to time) linear equation, the second one is a first order linear system of equations, the third one is a second order linear system while the last ones are first order non-linear systems. We shall demonstrate in this paper, that the problem of imposing boundary conditions for these systems of equations in finite volume setup can be handled in a systematic and effective way thanks to their hyperbolic character (which is dealt in the forthcoming Section).

2.2 On the continuous system of equations

The dependent variable v is a function of the independent variable x and t. Given an arbitrary point $\underline{v} \in G$, the constant function $v(x,t) \equiv \underline{v}$ solves the system (2.1). Now if we linearize this system around this trivial solution, we obtain :

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{nd} \underline{A^j} \frac{\partial w}{\partial x_j} = 0\,, \tag{2.22}$$

where $\underline{A^j} = A^j(\underline{v})$ and where $A^j(v)$ is the Jacobian matrix $\frac{\partial F^j(v)}{\partial v}$. Indeed, this readily follows from the first order Taylor expansion with respect to $w \in \mathbb{R}^{nd}$:

$$F_p^j(\underline{v}+w) = F_p^j(\underline{v}) + \sum_{k=1}^{m} \frac{\partial F_p^j(\underline{v})}{\partial v_k} w_k + \text{ h.o.t. in } w\,.$$

Now, equation (2.22) can be explicitly solved thanks to the Fourier transform which actually amounts looking for its plane wave solutions $w(x,t) = \xi \exp i(\omega.x - \lambda t)$ with $\omega \in \mathbb{R}^{nd}$ and $\lambda \in \mathbb{R}$. This leads to the following eigenvalue problem :

$$\sum_{j=1}^{nd} \left(\omega_j \underline{A^j}\right) \xi = \lambda\, \xi\,. \tag{2.23}$$

By denoting the $m \times m$ matrix

$$A_\omega(v) = \sum_{j=1}^{nd} \omega_j \frac{\partial F^j(v)}{\partial v}\,, \tag{2.24}$$

we see that we are dealing with the eigenvalue problem for $A_\omega(\underline{v})$. It is now natural to introduce the classical definition of hyperbolicity for systems like (2.1).

Definition 1 *The multidimensional system of conservation laws (2.1) is said to be hyperbolic if for every $\omega \in \mathbb{R}^{nd}$ and every $v \in G$ the $m \times m$ matrix $A_\omega(v)$ has m linearly independent real eigenvectors.*
In case of an hyperbolic system, an eigensystem of $A_\omega(v)$ is composed of

- *the set of the real eigenvalues :* $\lambda_1(v,\omega) \leq \ldots \leq \lambda_m(v,\omega)$,
- *a set* $(l_1(v,\omega), \ldots, l_m(v,\omega))$ *of left eigenvectors satisfying :*

$$^t A_\omega(v)\, l_k(v,\omega) = \lambda_k(v,\omega) l_k(v,\omega), \quad \text{for } k = 1,\ldots,m,$$

- *a set* $(r_1(v,\omega), \ldots, r_m(v,\omega))$ *of right eigenvectors satisfying :*

$$A_\omega(v)\, r_k(v,\omega) = \lambda_k(v,\omega) r_k(v,\omega), \quad \text{for } k = 1,\ldots,m.$$

and the following normalization : ($k, p = 1, \ldots, m$)

$$l_k(v,\omega) \cdot r_p(v,\omega) = \delta_{k,p}\,.$$

Let us discuss the hyperbolic character of the previously defined models and for a later use, let us give the analytical expression of an eigensystem.

2.2.1 Example 1 : Eigensystem

For the the multidimensional wave equation (2.2), given $\omega \in \mathbb{R}^{nd}$, the matrix A_ω (we omit the dependence in v since for a linear equation the matrix $A_\omega(v)$ does not depend on v) is

$$A_\omega = \begin{pmatrix} 0 & c\omega_1 & \cdots & c\omega_{nd} \\ c\omega_1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ c\omega_{nd} & 0 & 0 & 0 \end{pmatrix}. \tag{2.25}$$

Since this matrix is symmetric, it has a complete set of eigenvectors and therefore (2.2) is an hyperbolic system of conservation laws. The eigenvalues of A_ω are as follows :

$$\begin{cases} \lambda_1(\omega) = -c|\omega|\,, \\ \lambda_2(\omega) = \cdots = \lambda_{nd}(\omega) = 0\,, \\ \lambda_{nd+1}(\omega) = c|\omega|\,. \end{cases} \tag{2.26}$$

The right and left eigenvectors associated to these eigenvalues can be taken equal to

$$\left\{\begin{array}{l} r_1(\omega) = l_1(\omega) = \frac{1}{\sqrt{2}}(1, \frac{-\omega}{|\omega|}), \\ \\ r_{nd+1}(\omega) = l_{nd+1}(\omega) = \frac{1}{\sqrt{2}}(1, \frac{\omega}{|\omega|}), \\ \\ r_j(\omega) = l_j(\omega) = (0, \Omega_j)\,, \quad j = 2, \ldots, nd\,, \end{array}\right. \tag{2.27}$$

where $\Omega_1, \cdots, \Omega_{nd-1}$ is an orthonormal basis of the hyperplane orthogonal to ω.

2.2.2 Example 2 : Eigensystem

Concerning Maxwell's system of equations (2.6)-(2.7), the matrix A_ω is compactly defined by the identity (2.8) :

$$A_\omega(D, B) = (-\frac{\omega}{\mu} \wedge B, \frac{\omega}{\epsilon} \wedge D)\,. \tag{2.28}$$

The eigenvalues of A_ω are as follows,

$$\left\{\begin{array}{l} \lambda_1(\omega) = \cdots = \lambda_{nd-1}(\omega) = -c|\omega|\,, \\ \\ \lambda_{nd}(\omega) = \lambda_{nd+1}(\omega) = 0\,, \\ \\ \lambda_{nd+2}(\omega) = \cdots = \lambda_{2nd}(\omega) = c|\omega|\,, \end{array}\right. \tag{2.29}$$

where $c > 0$ satisfies $c^2\epsilon\mu = 1$.
The right eigenvectors associated to these eigenvalues can be taken equal to

$$\left\{\begin{array}{l} r_j(\omega) = (\Omega_j, -\frac{\omega \wedge \Omega_j}{c\epsilon|\omega|}), \quad j = 1, \ldots, nd-1\,, \\ \\ r_{nd}(\omega) = (\omega, 0), \\ \\ r_{nd+1}(\omega) = (0, \omega), \\ \\ r_{nd+j+1}(\omega) = (\Omega_j, \frac{\omega \wedge \Omega_j}{c\epsilon|\omega|}), \quad j = 1, \ldots, nd-1\,, \end{array}\right. \tag{2.30}$$

where $\Omega_1, \cdots, \Omega_{nd-1}$ is an orthonormal basis of the hyperplane orthogonal to ω. The dual basis of the $(r_k(\omega))_{k=1,\ldots,nd+2}$ is then

$$\left\{\begin{array}{l} l_j(\omega) = \frac{1}{2}(\Omega_j, -\frac{\omega \wedge \Omega_j}{c\mu|\omega|}), \quad j = 1, \ldots, nd-1\,, \\ \\ l_{nd}(\omega) = (\omega, 0), \\ \\ l_{nd+1}(\omega) = (0, \omega), \\ \\ l_{nd+j+1}(\omega) = \frac{1}{2}(\Omega_j, \frac{\omega \wedge \Omega_j}{c\mu|\omega|}), \quad j = 1, \ldots, nd-1\,. \end{array}\right. \tag{2.31}$$

2.2.3 Example 3 : Eigensystem

Concerning linear elasticity system of equations (2.10), the matrix A_ω is defined by :

$$\begin{pmatrix} 0_{nd} & A_1 & \cdots & A_{nd} \\ \omega_1 \mathbb{1}_{nd} & 0_{nd} & \cdots & 0_{nd} \\ \vdots & \vdots & \vdots & \vdots \\ \omega_{nd} \mathbb{1}_{nd} & 0_{nd} & \cdots & 0_{nd} \end{pmatrix}, \tag{2.32}$$

where 0_{nd} and $\mathbb{1}_{nd}$ are the zero and identity matrices and where matrices A_i are defined with the non-zero column and row corresponding to column i and row i by

$$A_i = \frac{\mu}{\rho}\omega_1 \mathbb{1}_{nd} + \frac{\lambda}{\rho}\begin{pmatrix} \cdots & 0 & \omega_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & \omega_{nd} & 0 & \cdots \end{pmatrix} + \frac{\mu}{\rho}\begin{pmatrix} \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \\ \omega_1 & \cdots & \omega_{nd} \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \end{pmatrix} \tag{2.33}$$

The eigenvalues of A_ω are equal to :

$$\begin{cases} \lambda_1(\omega) = -\sqrt{\dfrac{\lambda+2\mu}{\rho}}|\omega|\,, \\ \lambda_2(\omega) = \cdots = \lambda_{nd}(\omega) = -\sqrt{\dfrac{\mu}{\rho}}|\omega|\,, \\ \lambda_{nd+1}(\omega) = \cdots = \lambda_{nd^2}(\omega) = 0\,, \\ \lambda_{nd^2+1}(\omega) = \cdots = \lambda_{nd(nd+1)-1}(\omega) = \sqrt{\dfrac{\mu}{\rho}}|\omega|\,, \\ \lambda_{nd(nd+1)}(\omega) = \sqrt{\dfrac{\lambda+2\mu}{\rho}}|\omega|\,. \end{cases} \tag{2.34}$$

2.2.4 Example 4 : Eigensystem

For the ideal magneto-hydrodynamics equations, the matrix $A_\omega(v)$ is

$$\begin{pmatrix} 0 & \omega & 0 & 0 \\ K\omega - u\cdot\omega u & u\otimes\omega - k\omega\otimes u + u\cdot\omega\mathbb{1} & k\omega & (1-k)\omega\otimes B - B\cdot\omega\mathbb{1} \\ -\dfrac{2u\cdot\omega|B|^2}{\rho} + (K-H)u\cdot\omega & (H+\dfrac{|B|^2}{\rho})\omega - k(u\cdot\omega)u & (1+k)u\cdot\omega & (1-k)u\cdot\omega B - B\cdot\omega u \\ \dfrac{B\cdot\omega u - B\cdot u\omega}{\rho} & \dfrac{B\otimes\omega - B\cdot\omega\mathbb{1}}{\rho} & 0 & u\cdot\omega\mathbb{1} \end{pmatrix}.$$

Following the process of Roe and Balsara [47], Barth [4], Powell [42], and Powell *et al* [43], we are able to give a generalized form of the eigenvalues and eigenvectors of this system. For $nd = 3$, the real eigenvalues are as follows :

$$\begin{cases} \lambda_1(v,\omega) = u\cdot\omega - |\omega|c_f \\ \lambda_2(v,\omega) = u\cdot\omega - |\omega|c_s \\ \lambda_3(v,\omega) = u\cdot\omega - c_a \\ \lambda_4(v,\omega) = \lambda_5(v,\omega) = u\cdot\omega \\ \lambda_6(v,\omega) = u\cdot\omega + c_a \\ \lambda_7(v,\omega) = u\cdot\omega + |\omega|c_s \\ \lambda_8(v,\omega) = u\cdot\omega + |\omega|c_f\,. \end{cases} \tag{2.35}$$

The eigenvalues λ_4 and λ_5 correspond to an entropy and a magnetic-flux wave, λ_3 and λ_6 correspond to a pair of Alfvén waves, λ_2 and λ_7 (resp. λ_1 and λ_8) correspond to a pair of slow (resp. fast) magneto-acoustic waves. Here the thermodynamic temperature has been denoted by T, the specific entropy by s, the total enthalpy by H, the speed of sound by c and then (note that according to the second principle of thermodynamics, we have

$\left(\frac{\partial p}{\partial s}\right)_\rho > 0, \left(\frac{\partial p}{\partial \rho}\right)_s > 0$)

$$\left\{\begin{array}{l} k = \dfrac{1}{\rho T}\left(\dfrac{\partial p}{\partial s}\right)_\rho\,, \quad c = \sqrt{\left(\dfrac{\partial p}{\partial \rho}\right)_s}\,, \quad H = e + \dfrac{1}{2}|u|^2 + \dfrac{p}{\rho}\,, \quad K = c^2 + k(|u|^2 - H)\,, \\ \\ c_a = \dfrac{B\cdot\omega}{\sqrt{\rho}} \\ \\ c_f = \sqrt{\dfrac{1}{2}\left(c^2 + \dfrac{|B|^2}{\rho} + \sqrt{\delta}\right)} \qquad , \qquad c_s = \sqrt{\dfrac{1}{2}\left(c^2 + \dfrac{|B|^2}{\rho} - \sqrt{\delta}\right)} \\ \\ \text{with} \quad \delta = \left(c^2 + \dfrac{|B|^2}{\rho}\right)^2 - \dfrac{4c^2(B\cdot\omega)^2}{\rho|\omega|^2}\,. \end{array}\right. \tag{2.36}$$

In order to describe the eigenvectors associated to these eigenvalues, we note Ω_1 the orthogonal vector to ω such that

$$\left\{\begin{array}{l} \Omega_1 \in \text{Span}(\omega, B)\,, \\ |\Omega_1| = |\omega|\,, \\ (\omega, \Omega_1, \omega\wedge\Omega_1) \text{ is direct}\,. \end{array}\right. \tag{2.37}$$

and in order to take into account the limit case $\omega \cdot B = 0$ and $w = \gamma B$, Roe and Balsara introduced the following positive parameters that are defined by

$$\alpha_f^2 = \frac{c^2 - c_s^2}{c_f^2 - c_s^2} \qquad , \qquad \alpha_s^2 = \frac{c_f^2 - c^2}{c_f^2 - c_s^2} \tag{2.38}$$

and that satisfy the following relationships

$$\left\{\begin{array}{l} \alpha_f^2 + \alpha_s^2 = 1 \\ \\ \alpha_f^2 c_f^2 + \alpha_s^2 c_s^2 = c^2 \\ \\ (\alpha_f\alpha_s)^2 = \dfrac{c^2(B\cdot\Omega_1)}{\rho\sqrt{\delta}|\omega|^2} \\ \\ \alpha_f^2 = \dfrac{1}{2}\left(1 + \dfrac{c^2 - \frac{|B|^2}{\rho}}{\sqrt{\delta}}\right) \quad , \quad \alpha_s^2 = \dfrac{1}{2}\left(1 - \dfrac{c^2 - \frac{|B|^2}{\rho}}{\sqrt{\delta}}\right) \end{array}\right. \tag{2.39}$$

Let us recall that if R (resp. L) is an right (resp. left) eigenvector of the system written in the physical variables (ρ, u, p, B) then the corresponding "conservative" eigenvector r (resp. l) is given by the relationship

$$r = \frac{\partial V}{\partial W} R \quad \text{and} \quad l = \left(\frac{\partial W}{\partial V}\right)^t L \tag{2.40}$$

where

$$\frac{\partial V}{\partial W} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u & \rho\mathbb{1} & 0 & 0 \\ H - \frac{c^2}{k} & \rho u & \frac{1}{k} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix} \tag{2.41}$$

and

$$\frac{\partial W}{\partial V} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-u}{\rho} & \frac{1}{\rho}\mathbb{1} & 0 & 0 \\ k(\frac{c^2}{k} - H + |u|^2) & -ku & k & -kB \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}\,. \tag{2.42}$$

For $nd = 3$, the right primitive eigenvalues associated to λ_3, λ_4, λ_5, λ_6 are equal to

$$\left\{\begin{array}{l} R_3(v,\omega) = (0, -\omega \wedge \Omega_1, 0, -\sqrt{\rho}\,\omega \wedge \Omega_1) \\ R_4(v,\omega) = (1,0,0,0) \\ R_5(v,\omega) = (0,0,0,\dfrac{\omega}{|\omega|}) \\ R_6(v,\omega) = (0, +\omega \wedge \Omega_1, 0, -\sqrt{\rho}\,\omega \wedge \Omega_1) \end{array}\right. \tag{2.43}$$

The primitive right eigenvectors of the magneto-acoustic waves are

$$\left\{\begin{array}{l} R_1(v,\omega) = \alpha_f(\rho, -c_f\dfrac{\omega}{|\omega|}, c^2\rho, 0) + \alpha_s c\sqrt{\rho}\,\mathrm{sgn}(B\cdot\Omega_1)(0, \dfrac{+B\cdot\omega}{\rho c_f|\omega|}\dfrac{\Omega_1}{|\omega|}, 0, \dfrac{\Omega_1}{|\omega|}) \\ R_2(v,\omega) = \alpha_s(\rho, -c_s\dfrac{\omega}{|\omega|}, c^2\rho, 0) - \alpha_f c\sqrt{\rho}\,\mathrm{sgn}(B\cdot\Omega_1)(0, \dfrac{+c_f\mathrm{sgn}(B\cdot\omega)}{c\sqrt{\rho}}\dfrac{\Omega_1}{|\omega|}, 0, \dfrac{\Omega_1}{|\omega|}) \\ R_7(v,\omega) = \alpha_s(\rho, +c_s\dfrac{\omega}{|\omega|}, c^2\rho, 0) - \alpha_f c\sqrt{\rho}\,\mathrm{sgn}(B\cdot\Omega_1)(0, \dfrac{-c_f\mathrm{sgn}(B\cdot\omega)}{c\sqrt{\rho}}\dfrac{\Omega_1}{|\omega|}, 0, \dfrac{\Omega_1}{|\omega|}) \\ R_8(v,\omega) = \alpha_f(\rho, +c_f\dfrac{\omega}{|\omega|}, c^2\rho, 0) + \alpha_s c\sqrt{\rho}\,\mathrm{sgn}(B\cdot\Omega_1)(0, \dfrac{-B\cdot\omega}{\rho c_f|\omega|}\dfrac{\Omega_1}{|\omega|}, 0, \dfrac{\Omega_1}{|\omega|}) \end{array}\right.$$

So as to minimize the rounding error and to treat the triple umbilic point, we rewrite c_s as

$$c_s = \frac{c}{|\omega| c_f}\sqrt{\frac{(B\cdot\omega)^2}{\rho}}\,, \tag{2.44}$$

and we propose to compute α_f and α_s in the following way : if $c^2 - \frac{|B|^2}{\rho}$ then $\alpha_f = \alpha_s = \frac{1}{\sqrt{2}}$ else first compute

$$\phi = \frac{1}{2}\arctan\left(\frac{2c|B\cdot\Omega_1|}{|\omega|\sqrt{\rho}(c^2 - \frac{|B|^2}{\rho})}\right) \tag{2.45}$$

then if $\phi \geq 0$ then $\alpha_f = \cos(\phi)$ and $\alpha_s = \sin(\phi)$ else if $\phi \leq 0$ then $\alpha_f = -\sin(\phi)$ and $\alpha_s = \cos(\phi)$.
The left primitive eigenvectors are given by

$$\left\{\begin{array}{l} L_1(v,\omega) = \dfrac{1}{2\alpha_f^2 c_f^2}\left(\alpha_f(0, -c_f\dfrac{\omega}{|\omega|}, \dfrac{1}{\rho}, 0) + \alpha_s c\,\mathrm{sgn}(B\cdot\Omega_1)(0, \dfrac{+B\cdot\omega}{\rho c_f|\omega|}\dfrac{\Omega_1}{|\omega|}, 0, \dfrac{\Omega_1}{|\omega|})\right) \\ L_2(v,\omega) = \dfrac{1}{2c^2}\left(\alpha_s(0, -c_s\dfrac{\omega}{|\omega|}, \dfrac{1}{\rho}, 0) - \dfrac{\alpha_f c\,\mathrm{sgn}(B\cdot\Omega_1)}{\sqrt{\rho}}(0, \dfrac{+c_f\sqrt{\rho}\,\mathrm{sgn}(B\cdot\omega)}{c}\dfrac{\Omega_1}{|\omega|}, 0, \dfrac{\Omega_1}{|\omega|})\right) \\ L_3(v,\omega) = \dfrac{1}{2|\omega|^4}(0, -\omega\wedge\Omega_1, 0, -\dfrac{1}{\sqrt{\rho}}\,\omega\wedge\Omega_1) \\ L_4(v,\omega) = (1,0,0,-\dfrac{1}{c^2}) \\ L_5(v,\omega) = (0,0,0,\dfrac{\omega}{|\omega|}) \\ L_6(v,\omega) = \dfrac{1}{2|\omega|^4}(0, +\omega\wedge\Omega_1, 0, -\dfrac{1}{\sqrt{\rho}}\,\omega\wedge\Omega_1) \\ R_7(v,\omega) = \dfrac{1}{2c^2}\left(\alpha_s(0, +c_s\dfrac{\omega}{|\omega|}, \dfrac{1}{\rho}, 0) - \dfrac{\alpha_f c\,\mathrm{sgn}(B\cdot\Omega_1)}{\sqrt{\rho}}(0, \dfrac{-c_f\sqrt{\rho}\,\mathrm{sgn}(B\cdot\omega)}{c}\dfrac{\Omega_1}{|\omega|}, 0, \dfrac{\Omega_1}{|\omega|})\right) \\ R_8(v,\omega) = \dfrac{1}{2\alpha_f^2 c_f^2}\left(\alpha_f(0, +c_f\dfrac{\omega}{|\omega|}, \dfrac{1}{\rho}, 0) + \alpha_s c\,\mathrm{sgn}(B\cdot\Omega_1)(0, \dfrac{-B\cdot\omega}{\rho c_f|\omega|}\dfrac{\Omega_1}{|\omega|}, 0, \dfrac{\Omega_1}{|\omega|})\right) \end{array}\right. \tag{2.46}$$

For $nd = 2$, the real real eigenvalues are defined by :

$$\left\{\begin{array}{l} \lambda_1(v,\omega) = u\cdot\omega - |\omega| c_f \\ \lambda_2(v,\omega) = u\cdot\omega - |\omega| c_s \\ \lambda_3(v,\omega) = \lambda_4(v,\omega) = u\cdot\omega \\ \lambda_5(v,\omega) = u\cdot\omega + |\omega| c_s \\ \lambda_6(v,\omega) = u\cdot\omega + |\omega| c_f \,. \end{array}\right. \tag{2.47}$$

and the eigenvectors correspond to the $nd = 3$ eigenvectors.
Hence the multidimensional magneto-hydrodynamics system is an hyperbolic system of conservation laws.

2.2.5 Example 5 : Eigensystem

For the multidimensional Euler equations (2.17)-(2.19), the matrix $A_\omega(v)$ is found to be equal to :

$$A_\omega(v) = \left(\begin{array}{ccc} 0 & \omega & 0 \\ K\omega - u\cdot\omega u & u\otimes\omega - k\omega\otimes u + u\cdot\omega\mathbb{1} & k\omega \\ (K-H)u\cdot\omega & H\omega - k(u\cdot\omega)u & (1+k)u\cdot\omega \end{array}\right) \tag{2.48}$$

where the thermodynamic temperature has been denoted by T, the specific entropy by s, the total enthalpy by H, the speed of sound by c and then (note that according to the second principle of thermodynamics, we have $\left(\frac{\partial p}{\partial s}\right)_\rho > 0, \left(\frac{\partial p}{\partial \rho}\right)_s > 0$)

$$k = \frac{1}{\rho T}\left(\frac{\partial p}{\partial s}\right)_\rho, \quad c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}, \quad H = e + \frac{1}{2}|u|^2 + \frac{p}{\rho}, \quad K = c^2 + k(|u|^2 - H)\,. \tag{2.49}$$

The eigenvalues of $A_\omega(v)$ are as follows,

$$\left\{\begin{array}{l} \lambda_1(v,\omega) = u\cdot\omega - |\omega| c\,, \\ \lambda_2(v,\omega) = \cdots = \lambda_{nd+1}(v,\omega) = u\cdot\omega\,, \\ \lambda_{nd+2}(v,\omega) = u\cdot\omega + |\omega| c\,. \end{array}\right. \tag{2.50}$$

The right eigenvectors associated to these eigenvalues can be taken equal to

$$\left\{\begin{array}{l} r_1(v,\omega) = (1, u - c\frac{\omega}{|\omega|}, H - u\cdot\frac{\omega}{|\omega|}c)\,, \\ r_{nd+2}(v,\omega) = (1, u + c\frac{\omega}{|\omega|}, H + u\cdot\frac{\omega}{|\omega|}c), \\ r_2(v,\omega) = (1, u, H - \frac{c^2}{k})\,, \\ r_3(v,\omega) = (0,\Omega_1, u\cdot\Omega_1), \ldots, r_{nd+1}(v,\omega) = (0,\Omega_{nd-1}, u\cdot\Omega_{nd-1})\,, \end{array}\right. \tag{2.51}$$

where $\Omega_1, \cdots, \Omega_{nd-1}$ is an orthonormal basis of the hyperplane orthogonal to ω. The dual basis of the $(r_k(v,\omega))_{k=1,\ldots,nd+2}$ is then

$$\left\{\begin{array}{l} l_1(v,\omega) = \frac{1}{2c^2}\left(K + \frac{u\cdot\omega}{|\omega|}c, -ku - \frac{\omega}{|\omega|}c, k\right)\,, \\ l_{nd+2}(v,\omega) = \frac{1}{2c^2}\left(K - \frac{u\cdot\omega}{|\omega|}c, -ku + \frac{\omega}{|\omega|}c, k\right)\,, \\ l_2(v,\omega) = \frac{k}{c^2}(H - |u|^2, u, -1)\,, \\ l_3(v,\omega) = (-u\cdot\Omega_1, \Omega_1, 0), \ldots, l_{nd+1}(v,\omega) = (-u\cdot\Omega_{nd-1}, \Omega_{nd-1}, 0)\,. \end{array}\right. \tag{2.52}$$

Hence the multidimensional Euler equations is an hyperbolic system of conservation laws.

2.2.6 Example 6, 7, 8 : Eigensystem

For the isentropic multidimensional Euler equations, the matrix $A_\omega(v)$ is

$$A_\omega(v) = \begin{pmatrix} 0 & \omega \\ c^2\omega - u\cdot\omega u & u\otimes\omega + u\cdot\omega\mathbb{1} \end{pmatrix}. \tag{2.53}$$

where the speed of sound is defined by $c = \sqrt{\frac{dp}{d\rho}}$. The eigenvalues are as follows

$$\begin{cases} \lambda_1(v,\omega) = u\cdot\omega - |\omega|c\,, \\ \lambda_2(v,\omega) = \cdots = \lambda_{nd}(v,\omega) = u\cdot\omega\,, \\ \lambda_{nd+1}(v,\omega) = u\cdot\omega + |\omega|c\,. \end{cases} \tag{2.54}$$

The right eigenvectors associated to these eigenvalues can be taken equal to

$$\begin{cases} r_1(v,\omega) = (1, u - c\frac{\omega}{|\omega|})\,, \\ r_{nd+1}(v,\omega) = (1, u + c\frac{\omega}{|\omega|}) \\ r_2(v,\omega) = (0,\Omega_1), \ldots, r_{nd}(v,\omega) = (0,\Omega_{nd-1}), \end{cases} \tag{2.55}$$

where $\Omega_1,\ldots,\Omega_{nd-1}$ is an orthonormal basis of the hyperplane orthogonal to ω. The dual basis of the $(r_k(v,\omega))_{k=1,\ldots,nd+1}$ is then

$$\begin{cases} l_1(v,\omega) = \frac{1}{2c}(c + \frac{u\cdot\omega}{|\omega|}, -\frac{\omega}{|\omega|})\,, \\ l_{nd+1}(v,\omega) = \frac{1}{2c}(c - \frac{u\cdot\omega}{|\omega|}c, \frac{\omega}{|\omega|})\,, \\ l_2(v,\omega) = (-u\cdot\Omega_1,\Omega_1), \ldots, l_{nd}(v,\omega) = (-u\cdot\Omega_{nd-1},\Omega_{nd-1})\,. \end{cases} \tag{2.56}$$

Concerning Application 7 one has simply to take into the previous formulas $c \equiv c_0$ while for Application 8 one takes $c \equiv \sqrt{gh}$.

2.2.7 Remark on the well posed character of the system

If we consider an hyperbolic system, for the linearized system (2.22) it is then straightforward to show using the Fourier transform that the initial value problem in the whole space is well posed in various functional spaces. Then by fixed point techniques, one can extend these results to the nonlinear system (2.1). We refer *e.g.* to the recent books by D. Serre [50] and C. Dafermos [11] for that purpose. However when one is interested in the mixed initial and boundary value problem, the situation is much more involved. In the linear case, much can be done while in the non-linear one very little is known.
Let us briefly give a flavor of the problem by considering the very simple example of the linear transport equation. Given $c \in \mathbb{R}^*$, the one dimensional linear transport equation (called the advection equation) reads :

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0\,. \tag{2.57}$$

Here hyperbolicity is automatic and if this equation is posed on a bounded interval, say $]a,b[$, one can prescribe only one of the two boundary values $u(a,t)$ or $u(b,t)$ depending on the sign of c. More precisely, one can only give the information that enters into the domain $]a,b[$: if $c>0$ one can give $u(a,t)$, while if $c<0$ one can give $u(b,t)$.
In the nonlinear case, the situation is more involved even in the one dimensional case since the "speed" c will depend on the solution.

2.3 The discrete system of equations

The computational domain Ω is taken to be a polygonal domain and it is decomposed in small volumes K (the so-called control volumes) such that $\Omega = \cup_{K\in\mathcal{T}} K$. We assume that the control volumes K are polyhedra such that the interior boundary is the union of hypersurfaces $K \cap L$ where L belongs to the set $\mathcal{N}(K) = \{L \in \mathcal{T}/L \neq K$ and $K \cap L$ has positive $(nd-1)$-measure$\}$[2].

2.3.1 The finite volume approach

In order to approximate the average of the solution on the control volume K which reads ($vol(K)$ denotes the nd-dimensional volume of K) :

$$v_K(t) \equiv \frac{1}{vol(K)} \int_K v(x,t)\, dx\,, \tag{2.58}$$

system (2.1) is integrated on K and leads to the following equations :

$$vol(K)\frac{dv_K}{dt} + F^{\nu}_{\partial K} = 0\,. \tag{2.59}$$

Hence the time evolution of $v_K(t)$ is governed by the normal flux on the boundary of K :

$$F^{\nu}_{\partial K}(t) = \int_{\partial K} F(v(\sigma,t))\cdot\nu(\sigma)\, d\sigma, \tag{2.60}$$

where ∂K is the boundary of K, $\nu(\sigma)$ the unit external normal on ∂K and $d\sigma$ denotes the $(nd-1)$-volume element on this hypersurface.
The heart of the matter in finite volume methods consists in providing a formula for the normal fluxes $F^{\nu}_{\partial K}$ in terms of the $\{v_L\}_{L\in\mathcal{T}}$. We decompose the normal flux (2.60) into a sum :

$$F^{\nu}_{\partial K} = \sum_{L\in\mathcal{N}(K)} F_{K,L}\,, \tag{2.61}$$

where the unit normal on $K \cap L$ denoted by $\nu_{K,L}$ points into L and where :

$$F_{K,L} = \int_{K\cap L} F(v(\sigma,t))\cdot\nu_{K,L}\, d\sigma\,. \tag{2.62}$$

Since our goal is to approximate (2.59) by an ordinary differential equation for the $v_K(t)$'s, we have to express the normal flux (2.62) in terms of the $\{v_M\}_{M\in\mathcal{T}}$. In general, (2.59) is strongly dominated by finite speed nonlinear wave propagation phenomena and therefore we are led to consider a formula which uses a finite stencil. For practical reasons, including CPU and storage costs, it turns out that using the two neighboring values v_K and v_L is enough at least for first order approximations. This means that we are going to look for a formula that reads as :

$$F_{K,L} \approx area(K\cap L)\,\Phi(v_K, v_L; K, L)\,, \tag{2.63}$$

where Φ is the numerical flux to be constructed and $area(K\cap L)$ stands for the $(nd-1)$-dimensional volume of the hypersurface $K \cap L$.

At this level of generality the numerical flux has to satisfy two properties :

(i) consistency : $\Phi(w,w;K,L) = F(w)\cdot\nu_{K,L}$,

(ii) conservation : $\Phi(v,w;K,L) = -\Phi(w,v;L,K)$.

We refer to Section 2.3.2 concerning formulas that will be used in the numerical computations. But once a formula has been chosen, the semi discretization of (2.1) is a system of o.d.e.'s that reads :

$$vol(K)\frac{dv_K}{dt} + \sum_{L\in\mathcal{N}(K)} area(K\cap L)\Phi(v_K, v_L; K, L) = 0\,. \tag{2.64}$$

[2] At this stage, we intentionally ignore control volumes K which meet the boundary of Ω since it is the matter of Section 3 to deal with this question.

2.3.2 Numerical fluxes

Here we have to provide a formula for the numerical flux $\Phi(v_K, v_L; K, L)$ at the interface between two control volumes K and L. Since this vector is an approximation to the normal flux (2.62), a natural choice would be *e.g.*

$$\Phi(v_K, v_L; K, L) = \frac{F(v_K) + F(v_L)}{2} \cdot \nu_{K,L} \,, \tag{2.65}$$

but as it is well known and understood, this leads to unstable schemes. Actually this flux has to take into account the direction of propagation of information between the two volumes during the time step. This is the basis for the upwind schemes and we can cite among them Godunov scheme, Roe scheme (see Godlewski and Raviart [28] and reference therein) and the VFFC scheme from Ghidaglia *et al* [26] and [27]. The last two schemes belong to the family of flux schemes according to the following definition (Ghidaglia [25]).

Definition 2 *The numerical flux $\Phi(v, w; K, L)$ corresponds to a flux scheme when there exists a matrix $U(v, w; K, L)$ such that*

$$\Phi(v, w; K, L) = \frac{F(v) + F(w)}{2} \cdot \nu_{K,L} - U(v, w; K, L) \frac{F(w) - F(v)}{2} \cdot \nu_{K,L} \,. \tag{2.66}$$

In all the numerical computations, we shall use the VFFC scheme (Ghidaglia *et al* [26]) which corresponds to the following formula.

Definition 3 *The numerical flux of the "VFFC" method is obtained by formula (2.66) when we take :*

$$U(v, w; K, L) = sgn\left(A_{\nu_{K,L}}(\mu(v, w; K, L))\right) \,, \tag{2.67}$$

where $\mu(v, w; K, L)$ is a mean between v_K and v_L which only depends on the geometry of K and L, e.g. :

$$\mu(v, w; K, L) = \frac{vol(K)v + vol(L)w}{vol(K) + vol(L)} \,. \tag{2.68}$$

Remark 1 *When a matrix M has a complete set of eigenvectors, $sgn(M)$ is the matrix which has the same eigenvectors as M but whose eigenvalues are the sign ($\in \{-1, 0, +1\}$) of those of M.*

2.3.3 Time discretization

Although the system (2.1) has also a conservative structure with respect to time t, since this variable is one dimensional, it is more standard to use finite difference schemes for its time discretization.
When the explicit first order backward Euler's scheme is used for the total discretization of (2.64), one gets :

$$v_K^{n+1} = v_K^n - \frac{\Delta t_n}{vol(K)} \sum_{L \in \mathcal{N}(K)} area(K \cap L) \Phi(v_K^n, v_L^n; K, L) \,. \tag{2.69}$$

Of course this scheme is stable under a stringent C.F.L. condition, which is not the case for the implicit first order forward Euler's scheme :

$$v_K^{n+1} = v_K^n - \frac{\Delta t_n}{vol(K)} \sum_{L \in \mathcal{N}(K)} area(K \cap L) \Phi(v_K^{n+1}, v_L^{n+1}; K, L) \,. \tag{2.70}$$

Both schemes are first order consistent, as it is the case for the space discretization.

3 Discretization of the boundary conditions

Let K be a control volume which meets the boundary $\partial\Omega$. Of course formula (2.64) is no longer valid. Indeed when K meets the boundary of Ω on the face $K \cap \Omega$, we have to find the numerical flux $\Phi(v_K, K, \partial\Omega)$ that approximates the integral

$$F_{K,\partial\Omega} = \int_{K \cap \partial\Omega} F(v(\sigma, t)) \cdot \nu_K \, d\sigma. \tag{3.1}$$

Here ν_K denotes (instead of $\nu_{K,\partial\Omega}$) the unit normal to the face that points outside Ω. Then the finite volume scheme yields for such K :

$$\begin{aligned} vol(K)\frac{dv_K}{dt} + \sum_{L \in \mathcal{N}(K)} area(K \cap L) \Phi(v_K, v_L; K, L) \\ + \sum_{K \cap \partial\Omega} area(K \cap \partial\Omega) \Phi(v_K, K, \partial\Omega) \; = 0. \end{aligned} \tag{3.2}$$

In practice, this flux is not given by the physical boundary conditions and moreover, in general, (2.1) is an ill-posed problem if we try to impose either v or $F(v)\cdot\nu_K$ on $\partial\Omega$. This can simply be understood by analyzing the characteristics. Let us consider the linearization of the system around the state $\underline{v}$ and its projection on the normal direction to the face :

$$\frac{\partial v}{\partial t} + \underline{A}_{\nu_K}\frac{\partial v}{\partial \nu} = 0\,, \tag{3.3}$$

where $\frac{\partial v}{\partial \nu} = \nabla v\cdot\nu_K$ and where $\underline{A}_{\nu_K}$ is the advection matrix (see (2.24)) :

$$\underline{A}_{\nu_K} \equiv A_{\nu_K}(\underline{v})\,. \tag{3.4}$$

Since (2.1) is assumed to be hyperbolic, the matrix $\underline{A}_{\nu_K}$ is diagonalizable on $\mathbb{R}$ and by a change of coordinates, this system has the form of an uncoupled set of m advection equations :

$$\frac{\partial \eta_k}{\partial t} + \lambda_k\frac{\partial \eta_k}{\partial \nu} = 0\,,\quad k=1,\dots,m\,. \tag{3.5}$$

Here the $\lambda_k = \lambda_k(\underline{v},\nu_K)$ are the eigenvalues of $\underline{A}_{\nu_K}$ and according to the sign of these numbers, waves are going either into the domain Ω ($\lambda_k<0$) or out of the domain Ω ($\lambda_k>0$). Hence we expect that it is only possible to impose χ conditions on $K\cap\partial\Omega$ where $\chi \equiv \sharp\{k\in\{1,\dots,m\} \text{ such that } \lambda_k<0\}$.

We denote by x the coordinate along the outer normal, then system (3.3) reads :

$$\frac{\partial v}{\partial t} + \underline{A}_{\nu_K}\frac{\partial v}{\partial x} = 0\,, \tag{3.6}$$

which happens to be the linearization of the 1-dimensional (*i.e.* when $nd=1$) system.

At this point there are two different situations. The first one (which is termed as the non characteristic case in the literature) refers to the case where the matrix $\underline{A}_{\nu_K}$ is invertible, while the second one refers to the complementary case.

3.1 The non characteristic case

We label the eigenvalues $\lambda_k(\underline{v},\nu_K)$ of $\underline{A}_{\nu_K}$ by increasing order :

$$\lambda_1(\underline{v},\nu_K)\le\lambda_2(\underline{v},\nu_K)\le\ldots\le\lambda_\chi(\underline{v},\nu_K)<0<\lambda_{\chi+1}(\underline{v},\nu_K)\ldots\le\lambda_m(\underline{v},\nu_K)\,.$$

3.1.1 The case $\chi = 0$

In this case, all the information comes from inside Ω and therefore we take :

$$\Phi(v_K,K,\partial\Omega) = F(v_K)\cdot\nu_K\,. \tag{3.7}$$

In the Computational Fluid Dynamics literature, this is known as the "supersonic outflow" case.

3.1.2 The case $\chi = m$

In this case, all the information comes from outside Ω and therefore we take :

$$\Phi(v_K,K,\partial\Omega) = \Phi_{given}\,, \tag{3.8}$$

where Φ_{given} is the flux computed from the given physical boundary conditions. In the Computational Fluid Dynamics literature, this is known as the "supersonic inflow" case.

3.1.3 The case $1\le\chi\le m-1$

As already discussed in the introduction of this Section, we need χ scalar information coming from outside of Ω. Hence we assume that we have on physical ground χ relations on the boundary :

$$g_k(v) = 0\,,\quad k=1,\dots,\chi\,. \tag{3.9}$$

Remark 2 *The notation $g_k(v)=0$ means that we have a relation between the components of v. However, in general, the function g_k is not given explicitly in terms of v. In Example 2.1.5, for Euler equations, $g_k(v)$ could be the pressure which is not one of the components of v.*

Since we have to determine the m components of $\Phi(v_K, K, \partial\Omega)$, we need $m-\chi$ supplementary scalar information. Let us write them as

$$h_k(v) = 0\,, \quad k = \chi+1, \dots, m. \tag{3.10}$$

In general conditions (3.9) are named as "physical boundary conditions" while conditions (3.10) are named as "numerical boundary conditions".
Then we take :

$$\Phi(v_K, K, \partial\Omega) = F(v)\cdot\nu_K\,, \tag{3.11}$$

where v is solution to (3.9)-(3.10) (see however Remark 18 and Section 8 for a practical point of view).

Remark 3 *The system (3.9)-(3.10) for the m unknowns $v \in G$ is a $m \times m$ nonlinear system of equations. We are going to study its solvability in Theorem 1.*

Let us now first discuss the numerical boundary conditions (3.10). The $m-\chi$ supplementary information we need, come from inside of Ω. A natural idea is then to use the advection equation (3.5) which is, of course, a first order approximation of the nonlinear equation (2.1), with $\eta_k \equiv l_k(\underline{v}, \nu_K)\cdot v$ and $\lambda_k \equiv \lambda_k(\underline{v}, \nu_K)$.
Since for $k \geq \chi+1$ we have $\lambda_k(\underline{v}, \nu_K) > 0$, we know that $l_k(\underline{v}, \nu_K)\cdot v(x,t)$ on the boundary (ie for $x \in \partial\Omega$) at time t depends on the values of $l_k(\underline{v}, \nu_K)\cdot v$ inside Ω. This suggests that the so-called characteristic boundary conditions :

$$l_k(\underline{v}, \nu_K)\cdot v = l_k(\underline{v}, \nu_K)\cdot v_K\,, \quad k = \chi+1, \dots, m\,, \tag{3.12}$$

are good candidates for (3.10). Actually we prefer a slightly different boundary condition that reads :

$$l_k(\underline{v}, \nu_K)\cdot(F(v)\cdot\nu_K) = l_k(\underline{v}, \nu_K)\cdot(F(v_K)\cdot\nu_K)\,, \quad k = \chi+1, \dots, m\,. \tag{3.13}$$

In fact (3.12) was derived with $\underline{v} = v_K$ in the context of finite differences (we refer to the book of Hirsch [32] Chapter 19). We think, and this is also confirmed by our numerical experience, that (3.13) is more adapted to the finite volume approach since the unknown is the normal flux $F(v)\cdot\nu_K$ on the boundary $K \cap \partial\Omega$.
Hence the $m-\chi$ supplementary information (3.10) are settled to :

$$h_k(v) \equiv l_k(\underline{v}, \nu_K)\cdot(F(v)\cdot\nu_K) - l_k(\underline{v}, \nu_K)\cdot(F(v_K)\cdot\nu_K)\,, \quad k = \chi+1, \dots, m. \tag{3.14}$$

We are going to prove the following result on the solvability of (3.9)-(3.10).

Theorem 1 *In the case where $1 \leq \chi \leq m-1$, (recall that we are in the non characteristic case : $\lambda_k(\underline{v}, \nu_K) \neq 0$, $k = 1, \dots, m$) and*

$$det_{1\leq k,l\leq\chi}\left(\sum_{i=1}^{m} r_l^i(\underline{v}, \nu_K)\frac{\partial g_k}{\partial v_i}(\underline{v})\right) \neq 0\,; \tag{3.15}$$

with the choice (3.14) for the relations $h_k(v)$, the nonlinear system (3.9)-(3.10) has one and only one solution v, for $v - \underline{v}$, $g_k(\underline{v})$ and $h_k(\underline{v})$ sufficiently small.

Remark 4 *It follows immediately from (3.15) that the functions g_k are functionally independent.*

Proof We take the fixed vectors $r_l(\underline{v}, \nu_K) \equiv (r_l^1(\underline{v}, \nu_K), \dots, r_l^m(\underline{v}, \nu_K))$ for representing the difference between the variables v and $\underline{v}$:

$$v = \underline{v} + \sum_{l=1}^{m} \xi_l r_l(\underline{v}, \nu_K)$$

and we denote by $L(\cdot)$ the nonlinear function from $\mathbb{R}^m$ into itself defined by :

$$L(\xi) \equiv \begin{pmatrix} g_1(\underline{v} + \sum_{l=1}^{m} \xi_l r_l(\underline{v}, \nu_K)) \\ \dots \\ g_\chi(\underline{v} + \sum_{l=1}^{m} \xi_l r_l(\underline{v}, \nu_K)) \\ h_{\chi+1}(\underline{v} + \sum_{l=1}^{m} \xi_l r_l(\underline{v}, \nu_K)) \\ \dots \\ h_m(\underline{v} + \sum_{l=1}^{m} \xi_l r_l(\underline{v}, \nu_K)) \end{pmatrix}.$$

Let us then compute the partial derivatives of L with respect to the ξ_l :

$$\frac{\partial L_k}{\partial \xi_l}_{|\xi=0} = \sum_{i=1}^{m} r_l^i(\underline{v}, \nu_K)\frac{\partial g_k}{\partial v_i}(\underline{v})\,, \quad k = 1, \dots, \chi\,, \tag{3.16}$$

$$\frac{\partial L_k}{\partial \xi_l}_{|\xi=0} = \lambda_k(\underline{v}, \nu_K)\delta_{k,l}\,, \quad k = \chi+1, \ldots, m\,. \tag{3.17}$$

Now thanks to condition (3.15) and the fact that $\lambda_k(\underline{v}, \nu_K) \neq 0$, for $k = \chi+1, \ldots, m$, we see that the Jacobian matrix of L is invertible at the point $\xi = 0$. On the other hand, since the size of $L(0)$ is small when the $g_k(\underline{v})$ and $h_k(\underline{v})$ are small, we can apply the local inversion theorem to conclude the proof of Theorem 1. ■

Remark 5 *If we take instead of (3.14),*

$$\tilde{h}_k(v) \equiv l_k(\underline{v}, \nu_K)\cdot v - l_k(\underline{v}, \nu_K)\cdot v_K\,, \quad k = \chi+1, \ldots, m \tag{3.18}$$

we have the following result.

Theorem 2 *In the case where $1 \leq \chi \leq m-1$, (recall that we are in the non characteristic case : $\lambda_k(\underline{v}, \nu_K) \neq 0$, $k = 1, \ldots, m$) and*

$$det_{1\leq k,l\leq \chi}\left(\sum_{i=1}^{m} r_l^i(\underline{v}, \nu_K)\frac{\partial g_k}{\partial v_i}(\underline{v})\right) \neq 0\,; \tag{3.19}$$

with the choice (3.18) instead of (3.14) for the relations $h_k(v)$, the nonlinear system (3.9)-(3.10) has one and only one solution v, for $v - \underline{v}$, $g_k(\underline{v})$ and $h_k(\underline{v})$ sufficiently small.

Remark 6 *Now we have to discuss the choice of $\underline{v}$. From Theorem 1, $g_k(\underline{v})$ for $k = 1, \ldots, \chi$ and $h_k(\underline{v})$ for $k = \chi+1, \ldots, m$ have to be "sufficiently small". Therefore the best choice should be the state v, solution to the system (3.9)-(3.10) for which g_k and h_k are zero. But since the computation of the eigenelements of the matrix $\underline{A}_{\nu_K}$ may be complex and the system (3.9)-(3.10) may be even more nonlinear, in practice an "easily computed" approximation of v is used. Usually $\underline{v}$ is taken equal to the interior state : $\underline{v} = v_K$. However, from a practical point of view, it might be possible to use other approximations : see for instance the case of the boundary conditions at infinity in Section 3.4.*

Remark 7 *If the condition (3.15) guarantees the existence of a solution to (3.9)-(3.10), it does not necessarily guarantee that the discrete solution of the hyperbolic system will be an admissible solution i.e. will satisfy any entropy admissibility condition.*

3.2 The characteristic case

If $n_0 \geq 1$ denotes the dimension of the kernel of the matrix $\underline{A}_{\nu_K}$ appearing in (3.3), we label its eigenvalues as follows :

$$\lambda_1(\underline{v}, \nu_K) \leq \lambda_2(\underline{v}, \nu_K) \leq \ldots \leq \lambda_\chi(\underline{v}, \nu_K) < 0 < \lambda_{\chi+n_0+1}(\underline{v}, \nu_K) \ldots \leq \lambda_m(\underline{v}, \nu_K)\,,$$
$$\lambda_{\chi+1}(\underline{v}, \nu_K) = \ldots = \lambda_{\chi+n_0}(\underline{v}, \nu_K) = 0\,.$$

3.2.1 The case $\chi = 0$

In this case, we simply follow the non characteristic case where we consider that all the information comes from inside and we take again (3.7)

$$\Phi(v_K, K, \partial\Omega) = F(v_K)\cdot\nu_K\,. \tag{3.20}$$

3.2.2 The case $\chi + n_0 = m$

Again we simply follow the non characteristic case where we consider that all the information comes from outside and we take again (3.8)

$$\Phi(v_K, K, \partial\Omega) = \Phi_{given}\,. \tag{3.21}$$

3.2.3 The case $1 \leq \chi \leq m - n_0 - 1$

We have χ scalar information coming from outside of Ω. Hence we assume that we have, on the physical ground, χ relations on the boundary :

$$g_k(v) = 0\,, \quad k = 1, \ldots, \chi\,. \tag{3.22}$$

Since we have to determine the m components of $\Phi(v_K, K, \partial\Omega)$, we need $m - \chi$ supplementary scalar information. Let us first write $m - \chi - n_0$ conditions according to what we have done in the non characteristic case :

$$l_k(\underline{v}, \nu_K)\cdot(F(v)\cdot\nu_K) = l_k(\underline{v}, \nu_K)\cdot(F(v_K)\cdot\nu_K)\,, \quad k = \chi+n_0+1, \ldots, m\,. \tag{3.23}$$

At first sight, we need n_0 supplementary information to determine the m components of the normal flux $F(v)\cdot\nu_K$. But since the mapping $v \to F(v)\cdot\nu_K$ has a non invertible Jacobian for $v = \underline{v}$, it happens that it is not true. For instance in the case of the Euler equations, for the wall boundary condition, since the relation (3.22) simply reads $u\cdot\nu_K = 0$, the normal flux $F(v)\cdot\nu_K = (0, p\nu_K, 0)$ depends on only one variable, the pressure p which may indeed be determined with relation (3.23). We refer to Section 6.1 for full details.

In the linear case, the situation is much simpler since $F(v)\cdot\nu_K = \underline{A}_{\nu_K} v$ has exactly $m - n_0$ independent components and there is only one numerical flux $F(v)\cdot\nu_K$ satisfying (3.22)-(3.23) provided again condition (3.15) holds true. Indeed the proof of Theorem 1 holds if the difference between the variables v and $\underline{v}$ is replaced by

$$v - \underline{v} = \sum_{l=1}^{\chi} \xi_l r_l(\underline{v}, \nu_K) + \sum_{l=\chi+n_0+1}^{m} \xi_l r_l(\underline{v}, \nu_K).$$

Let us remark that v solution of (3.22)-(3.23) is no longer unique. We refer to Section 5.1 for the multidimensional wave equation and to Section 5.2 for Maxwell's equations.

3.3 Riemann invariants and boundary conditions treatment

3.3.1 The Riemann problem and pseudo Riemann invariants

As it is well known, the Riemann problem for one dimensional hyperbolic equations has played a very important role for the numerical computations of its solutions. Godunov [29] has based his famous method on the explicit solution to this problem and later Roe [46] has simplified a lot Godunov's method by introducing a relevant linear problem at each interface for which (like for any linear hyperbolic system in one space dimension) the Riemann problem is straightforward to compute. Up to now, this strategy has not been generalized to higher space dimensions ; and there are profound reasons for that : even on Cartesian meshes and for linear equations, the solution to the Riemann problem is not simple to obtain and no closed formulas are available. Nevertheless, multidimensional problems can benefit from one dimensional ones by considering at each interface the normal equation, that is :

$$\frac{\partial v}{\partial t} + \frac{\partial F(v)\cdot\nu_K}{\partial \nu} = 0\,. \tag{3.24}$$

Here ν_K denotes a unitary normal vector on the interface and ν a coordinate in the direction of ν_K. Having fixed this interface, we denote by $f(v) \equiv F(v)\cdot\nu_K$ and by x the previous coordinate so that we are in the one dimensional setting :

$$\frac{\partial v}{\partial t} + \frac{\partial f(v)}{\partial x} = 0\,. \tag{3.25}$$

We denote here by $A(v)$ the Jacobian matrix $\frac{\partial f(v)}{\partial v}$. We assume that (3.25) is hyperbolic (see Definition 1) and introduce an eigensystem of $A(v)$ composed of

- the set of the real eigenvalues : $\lambda_1(v) \le \ldots \le \lambda_m(v)$,
- a set $(l_1(v), \ldots, l_m(v))$ of left eigenvectors satisfying :

$$^t A(v)\, l_k(v) = \lambda_k(v) l_k(v), \quad \text{for } k = 1, \ldots, m,$$

- a set $(r_1(v), \ldots, r_m(v))$ of right eigenvectors satisfying :

$$A(v)\, r_k(v) = \lambda_k(v) r_k(v), \quad \text{for } k = 1, \ldots, m.$$

  and the following normalization : $(k, p = 1, \ldots, m)$

$$l_k(v)\cdot r_p(v) = \delta_{k,p}\,.$$

For these one dimensional systems we have the classical notion of Riemann invariants (*see* Definition 4). Actually this definition can be introduced for a general hyperbolic system of equations that reads as :

$$\frac{\partial v}{\partial t} + A(v)\frac{\partial v}{\partial x} = 0\,. \tag{3.26}$$

In other words, the conservative structure of (3.25) is not needed and we shall have in mind to deal with regular solutions.

Definition 4 *A function W from G into $\mathbb{R}$ is a Riemann invariant for (3.26) provided that for every $v \in G$, the vector $\nabla W(v)$ is a left eigenvector of $A(v)$: $\exists\lambda$: $G \to \mathbb{R}$ such that*

$$^t A(v)\,\nabla W(v) = \lambda(v)\nabla W(v)\,. \tag{3.27}$$

Remark 8 *Let W be a Riemann invariant. If $\nabla W(v)$ is parallel to one of the $l_p(v)$ (this will always be the case when the eigenvalues are simple) then for every $k \neq p$ we have :*

$$r_k(v) \cdot \nabla W(v) = 0\,. \tag{3.28}$$

In the case where one can find m independent[3] Riemann invariants $W_1, \ldots, W_m$, it is possible to diagonalize (3.26) :

$$\frac{\partial W_k(v)}{\partial t} + \lambda_k(v)\frac{\partial W_k(v)}{\partial x} = 0\,, \tag{3.29}$$

(this is only valid for regular solutions). On the one hand, it is not possible, in general, to find m independent Riemann invariants : systems for which this is possible are "completely integrable systems". On the other hand, *linear* hyperbolic systems are always integrable since we can take

$$W_k(v) \equiv l_k \cdot v\,. \tag{3.30}$$

■

Let us return to the general case. Since it is not possible to find m independent functions satisfying (3.27), we are going to find $m-1$ functions satisfying (3.28). More precisely we first set the following definition.

Definition 5 *A function W from G into $\mathbb{R}$ is a k-pseudo Riemann invariant for (3.26) provided that for every $v \in G$, the vector $\nabla W(v)$ satisfies :*

$$r_k(v) \cdot \nabla W(v) = 0\,. \tag{3.31}$$

A classical result is then (see *e.g.* Smoller [51]) as follows.

Theorem 3 *Let $\underline{v}$ be given in G. For every $k \in \{1, \ldots, m\}$, there exists $\epsilon > 0$ and $m-1$ independent k-pseudo Riemann invariants in the set of those v for which $||v - \underline{v}|| \leq \epsilon$.*

Among the solutions of (3.25), the simple waves play a distinguished role.

Definition 6 *A weak solution $w(x,t)$ to (3.25) is called a k-simple wave provided that for every k-pseudo Riemann invariant W, $W(w(x,t))$ is constant with respect to x and t.*

These solutions are important because they are the building blocks of the solution to the Riemann problem associated with (3.25) (see again for instance Smoller [51]). Recall that the Riemann problem associated for (3.25) consists in solving this equation with the initial data :

$$v(x,0) = v_l \text{ for } x < 0 \quad \text{and} \quad v(x,0) = v_r \text{ for } x > 0\,. \tag{3.32}$$

As it is well known, the Riemann problem is central in the Godunov method where at each numerical interface one solves such a problem. Although this approach is known to be be somewhat costly (with regards to CPU) and only tractable for a very limited number of problems, we mention it because it happens to be used for finding boundary conditions with a nonlinear approach : see Section V.2 of Godlewski and Raviart [28] and references therein.
Recall that the solution to (3.25)-(3.32) is constructed by gluing m-simple waves. More precisely one finds $m-1$ constant states $v_1, \ldots, v_{m-1}$ and glues together m simple waves joining v_k to v_{k+1} for $k = 0, 1, \ldots, m-1$ with $v_0 = v_l$ and $v_m = v_r$. According to Definition 6, in order to effectively construct the solution to the Riemann problem, it is important to find the k-pseudo Riemann invariants.

Remark 9 *Let us observe that the k-pseudo Riemann invariants can be computed for any arbitrary system of variables. Indeed, if we change the dependent variables v, it will change the matrix $A(v)$ in (3.25) but pseudo Riemann invariants will not be affected ! This can be seen as follows. Assume that we take $\overline{v} = \varphi(v)$ a new set of variables in $\mathbb{R}^m$. Equation (3.26) reads now as :*

$$\frac{\partial \overline{v}}{\partial t} + \overline{A}(\overline{v})\frac{\partial \overline{v}}{\partial x} = 0\,, \tag{3.33}$$

where $\overline{A}(\overline{v}) \equiv J(v)A(v)J^{-1}(v)$ with $v = \varphi^{-1}(\overline{v})$ and $J(v) \equiv \frac{\partial \varphi(v)}{\partial v}$. Of course the right eigenvectors of $\overline{A}(\overline{v})$ can simply be taken as the $\overline{r}_k(\overline{v}) \equiv J(v)r_k(v)$ and for a given function $W(v)$, if one denotes by $\overline{W}(\overline{v}) \equiv W(\varphi^{-1}(\overline{v}))$ the corresponding function in the $\overline{v}$ variables, one can easily compute that

$$r_k(v) \cdot \nabla_v W(v) = \overline{r}_k(\overline{v}) \cdot \nabla_{\overline{v}} \overline{W}(\overline{v})\,.$$

In view of Definition 5, this proves our claim that pseudo Riemann invariants do not depend on the set of dependent variables we are using. An important consequence of that is that one can use "adapted" variables to solve the first order linear differential equation (3.31). This technique is successfully applied to the one-dimensional Euler system (see Section 10.1.5).

[3]Recall that m functions $W_1, \ldots, W_m$ are said to be independent if for every v the vectors $\nabla W_1(v), \ldots, \nabla W_m(v)$ are linearly independent.

3.3.2 Application to the boundary condition problem

At a boundary, it is not possible to solve the Riemann problem since the exterior state is not known. For instance, the strategy proposed by Dubois and LeFloch [20], Dubois [19], see also Buffard *et al* [7] for a linearized version, consists in solving an "incomplete" Riemann problem[4]. For a case where $1 \leq \chi \leq m-1$ *i.e.* where χ information are entering into the domain, this strategy solves (3.25) with the initial data :

$$\left\{ \begin{array}{c} v(x,0) = \underline{v} \text{ for } x < 0 \\ \text{and} \\ v(x,0) \text{ satisfies } \chi \text{ physical conditions for } x > 0. \end{array} \right. \tag{3.34}$$

Let us note the χ physical boundary conditions :

$$g_k(v) = 0\,, \quad k = 1, \ldots, \chi\,. \tag{3.35}$$

The "incomplete" Riemann problem method consists then in finding $\chi - 1$ intermediate states $\mu_1, \ldots, \mu_{\chi-1}$ such that $\underline{v}$ is connected to μ_1 through a 1-simple wave, μ_1 is connected to μ_2 through a 2-simple wave, ..., $\mu_{\chi-1}$ is connected to v through a χ-simple wave. Using Definition 6, we can rewrite these conditions, with $\mu_0 \equiv \underline{v}$ and $\mu_\chi \equiv v$, as :

$$W_k^\ell(\mu_{\ell-1}) = W_k^\ell(\mu_\ell)\,, k = 1, \ldots, m-1\,, \ell = 1, \ldots, \chi\,, \tag{3.36}$$

where $\{W_k^\ell\}_{k=1,\ldots,m-1}$ is a family of $m-1$ independent ℓ-pseudo Riemann invariants (in the vicinity of $\underline{v}$). The system (3.35)-(3.36) is made of $m \times \chi$ scalar equations for $m \times \chi$ scalar unknowns : the coordinates of $\mu_1, \ldots, \mu_\chi$ (provided the pseudo Riemann invariants have been computed). Figure 1 depicts the solution of the incomplete Riemann problem which comprises χ simple waves in the plane (x,t). More details concerning their values are given in Section 10 in the context of one dimensional Euler equations.

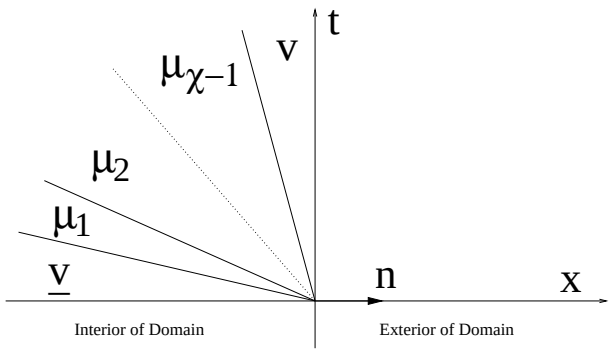

Figure 1: The plane (x,t) for the "incomplete" Riemann problem.

Let us emphasize that the method we propose in Section 3.1.3, replaces the system (3.36) by the system of $m - \chi$ equations :

$$l_k(\underline{v}) \cdot f(v) = l_k(\underline{v}) \cdot f(\underline{v})\,, \quad k = \chi + 1, \ldots, m. \tag{3.37}$$

Hence our method appears to be much simpler than the one that uses an incomplete Riemann problem. In a certain sense, system (3.37) solves system (3.36) for v. Actually, system (3.37) provides an approximation for the solution of (3.36). Let us study now the relationship between these two systems.

3.3.3 Comparison with our method

Setting. Let us, as a typical example, consider the case where $\chi = m - 1$ that is the case where $m - 1$ information are entering into the domain : this case occurs *e.g.* in the context of fluid dynamics for a subsonic inlet, see Section 10.1. Let us give a state $\underline{v}$ such that

$$\lambda_1(\underline{v}) \leq \ldots \leq \lambda_{m-1}(\underline{v}) < 0 < \lambda_m(\underline{v})\,, \tag{3.38}$$

and where 0 is not an eigenvalue (non characteristic case). Let us recall that for the "incomplete" Riemann method, the $m-2$ intermediates states $\mu_1, \ldots, \mu_{m-2}$ satisfy the following system made of $m \times (m-1)$ scalar equations with $\mu_0 \equiv \underline{v}$ and $\mu_{m-1} \equiv v$:

$$W_k^\ell(\mu_{\ell-1}) = W_k^\ell(\mu_\ell)\,, k = 1, \ldots, m-1\,, \ell = 1, \ldots, m-1\,, \tag{3.39}$$

[4]The terminology "partial" or "half" Riemann problem is also used.

and that our strategy replaces this system by the single equation :

$$l_m(\underline{v}) \cdot f(v) = l_m(\underline{v}) \cdot f(\underline{v}) \,. \tag{3.40}$$

Given $\epsilon > 0$, we introduce the three sets :

$$E_\epsilon = \left\{ \begin{array}{l} v \in G \,/ \\ \quad ||v - \underline{v}|| \leq \epsilon \text{ and } l_m(\underline{v}) \cdot f(v) = l_m(\underline{v}) \cdot f(\underline{v}) \end{array} \right\} , \tag{3.41}$$

$$F_\epsilon = \left\{ \begin{array}{l} (v, \mu_1, \ldots, \mu_{m-2}) \in G^{m-1} \,/ \\ \quad ||v - \underline{v}|| \leq \epsilon \,, ||\mu_i - \underline{v}|| \leq \epsilon \,, i = 1, \ldots, m-2 \,, \text{ and} \\ \quad W_k^\ell(\mu_{\ell-1}) = W_k^\ell(\mu_\ell) \,, k = 1, \ldots, m-1 \,, \ell = 1, \ldots, m-1 \end{array} \right\} , \tag{3.42}$$

$$G_\epsilon \equiv Pr_1(F_\epsilon) = \left\{ \begin{array}{l} v \in G \,/ \\ \quad \exists \mu = (\mu_1, \ldots, \mu_{m-2}) \in G^{m-1} \text{ with } (v, \mu) \in F_\epsilon \end{array} \right\} . \tag{3.43}$$

Clearly E_ϵ is related to our method for imposing the boundary conditions while F_ϵ and G_ϵ are related to the method relying on the Riemann invariants.

Relation between E_ϵ and G_ϵ. Our goal is to show the following result.

Theorem 4 *With the previous notations and under assumption (3.38), there exists ϵ_0 and two functions ϕ and $\tilde{\phi}$ defined respectively on neighborhoods $\mathcal{O}$ and $\tilde{\mathcal{O}}$ of 0 in $\mathbb{R}^{m-1}$ such that for every $\epsilon < \epsilon_0$ we have :*

$$E_\epsilon = \left\{ \underline{v} + \sum_{i=1}^{m-1} \alpha_i r_i(\underline{v}) + \phi(\alpha_1, \ldots, \alpha_{m-1}) r_m(\underline{v}) \,, (\alpha_1, \ldots, \alpha_{m-1}) \in \mathcal{O} \right\} , \tag{3.44}$$

$$G_\epsilon = \left\{ \underline{v} + \sum_{i=1}^{m-1} \beta_i r_i(\underline{v}) + \tilde{\phi}(\beta_1, \ldots, \beta_{m-1}) r_m(\underline{v}) \,, (\beta_1, \ldots, \beta_{m-1}) \in \tilde{\mathcal{O}} \right\} . \tag{3.45}$$

Moreover $\phi(0) = \tilde{\phi}(0) = 0$ and $\nabla\phi(0) = \nabla\tilde{\phi}(0) = 0$ i.e. the two sets are tangent at the point $\underline{v}$ to the hyperspace orthogonal to $l_m(\underline{v})$.

Proof i) Let us first characterize the set E_ϵ. For we represent all the points in G by the coordinates $\alpha \equiv (\alpha_1, \ldots, \alpha_m)$ as follows : $v = \underline{v} + \sum_{i=1}^m \alpha_i r_i(\underline{v})$. In order to rewrite (3.40) in these coordinates, first we write the Taylor expansion :

$$f(v) = f(\underline{v}) + A(\underline{v})(v - \underline{v}) + \int_0^1 (1-\theta) \frac{\partial A}{\partial v}(\theta v + (1-\theta)\underline{v})(v - \underline{v}, v - \underline{v}) d\theta \,,$$

and then we take the product scalar with $l_m(\underline{v})$ so that

$$l_m(\underline{v}) \cdot f(v) = l_m(\underline{v}) \cdot f(\underline{v}) + \lambda_m(\underline{v})\alpha_m + l_m(\underline{v}) \cdot \int_0^1 (1-\theta) \frac{\partial A}{\partial v}(\theta v + (1-\theta)\underline{v})(v - \underline{v}, v - \underline{v}) d\theta \,.$$

Introducing the function g defined on a neighborhood of 0 by the formula

$$\begin{array}{l} g(\alpha_1, \ldots, \alpha_m) \equiv \\ \quad l_m(\underline{v}) \cdot \displaystyle\int_0^1 (1-\theta) \frac{\partial A}{\partial v} \left(\underline{v} + \theta \sum_{i=1}^m \alpha_i r_i(\underline{v}) \right) \left(\sum_{i=1}^m \alpha_i r_i(\underline{v}), \sum_{i=1}^m \alpha_i r_i(\underline{v}) \right) d\theta \,, \end{array} \tag{3.46}$$

we see that (3.40) simply reads as

$$\lambda_m(\underline{v})\alpha_m + g(\alpha_1, \ldots, \alpha_m) = 0 \,. \tag{3.47}$$

Since g is flat at $\alpha = 0$, we have $g(0) = 0$ and $\dfrac{\partial g}{\partial \alpha_i}(0) = 0$ for all i. Hence, by application of the classical implicit function theorem, for small α, equation (3.47) has a unique solution $\alpha_m = \phi(\alpha_1, \ldots, \alpha_{m-1})$. Moreover by differentiating the equation

$$\lambda_m(\underline{v})\phi(\alpha_1, \ldots, \alpha_{m-1}) + g(\alpha_1, \ldots, \alpha_{m-1}, \phi(\alpha_1, \ldots, \alpha_{m-1})) = 0 \,, \tag{3.48}$$

with respect to α_i, $i = 1, \ldots, m-1$, we obtain that $\nabla\phi(0) = 0$. This shows the first part of Theorem 4.
ii) Concerning G_ϵ, we start with the study of F_ϵ and prove the following result.

Proposition 1 *With the previous notations and under assumption (3.38), there exists ϵ_0 and $m(m-1)$ functions $\psi_{i,\ell}$ defined on a neighborhood $]-\eta_\ell,+\eta_\ell[$ of 0 in $\mathbb{R}$ such that for every $\epsilon<\epsilon_0$ we have :*

$$F_\epsilon^\ell=\left\{\begin{array}{l}\mu_\ell=\mu_{\ell-1}+\displaystyle\sum_{i\neq\ell}\psi_{i,\ell}(\alpha_\ell)r_i(\mu_{\ell-1})+\alpha_\ell r_\ell(\mu_{\ell-1})\,/\\ \quad|\alpha_\ell|\leq\eta_\ell\end{array}\right\}, \tag{3.49}$$

where for $\ell=1,\dots,m-1$,

$$F_\epsilon^\ell\equiv\left\{\begin{array}{l}(\mu_{\ell-1},\mu_\ell)\in G^2\,/\\ \quad||\mu_{\ell-1}-\underline{v}||\leq\epsilon,||\mu_\ell-\underline{v}||\leq\epsilon\text{ and}\\ \quad W_k^\ell(\mu_{\ell-1})=W_k^\ell(\mu_\ell)\,,k=1,\dots,m-1\end{array}\right\}. \tag{3.50}$$

Moreover the functions $\psi_{i,\ell}$ are flat at $\alpha_\ell=0$ i.e. $\psi_{i,\ell}(0)=0$ and $\psi'_{i,\ell}(0)=0$.

Proof For ℓ fixed between 1 and $m-1$, we start with

$$W_k^\ell(\mu_{\ell-1})=W_k^\ell(\mu_\ell)\,,\quad k=1,\cdots,m-1, \tag{3.51}$$

and we write

$$\mu_\ell=\mu_{\ell-1}+\sum_{i=1}^m\alpha_i r_i(\mu_{\ell-1}). \tag{3.52}$$

Performing a Taylor expansion of the function $W_k^\ell(\mu_{\ell-1}+\sum_{i=1}^m\alpha_i r_i(\mu_{\ell-1}))$ with respect to α, we obtain :

$$W_k^\ell(\mu_\ell)=W_k^\ell(\mu_{\ell-1})+\sum_{i=1}^m\alpha_i r_i(\mu_{\ell-1})\cdot\nabla W_k^\ell(\mu_{\ell-1})+h_k^\ell(\alpha_1,\dots,\alpha_m)\,, \tag{3.53}$$

where the function h_k^ℓ is flat at $\alpha=0$. Hence (3.51) reads as :

$$\sum_{i=1}^m\alpha_i r_i(\mu_{\ell-1})\cdot\nabla W_k^\ell(\mu_{\ell-1})+h_k^\ell(\alpha_1,\dots,\alpha_m)=0\,. \tag{3.54}$$

But according to Definition 5, $r_\ell(\mu_{\ell-1})\cdot\nabla W_k^\ell(\mu_{\ell-1})=0$ so that we have in fact :

$$\sum_{i\neq\ell}\alpha_i r_i(\mu_{\ell-1})\cdot\nabla W_k^\ell(\mu_{\ell-1})+h_k^\ell(\alpha_1,\dots,\alpha_m)=0\,,\quad k=1\dots,m-1\,. \tag{3.55}$$

This is a system of $m-1$ equations for the m unknowns $(\alpha_i)_{1\leq i\leq m}$, but since the $m-1$ pseudo-Riemann invariants W_k^ℓ are independent, the determinant

$$det\left\{\begin{array}{l}i=1,\dots,m\,;\,i\neq\ell\\ k=1,\dots,m-1\end{array}\right.\quad\Big(r_i(\mu_{\ell-1})\cdot\nabla W_k^\ell(\mu_{\ell-1})\Big)$$

is different from zero (see Lemma 1 hereafter) and by the implicit function theorem, we can express all the α_i's for $i\neq\ell$ as functions of α_ℓ. That is (3.55) can be solved as

$$\mu_\ell=\mu_{\ell-1}+\sum_{i\neq\ell}\psi_{i,\ell}(\alpha_\ell)r_i(\mu_{\ell-1})+\alpha_\ell r_\ell(\mu_{\ell-1})\,,\quad|\alpha_\ell|\leq\eta_\ell\,. \tag{3.56}$$

Now since the functions h_k are flat at $\alpha=0$, we also deduce that $\psi_{i,\ell}(0)=0$ and $\psi'_{i,\ell}(0)=0$. And this achieves the proof of Proposition 1.
Then returning to that of Theorem 4, we write all the characterizations (3.49) :

$$\mu_\ell=\phi_\ell(\alpha_\ell,\mu_{\ell-1}),\quad\ell=1,\cdots,m-1. \tag{3.57}$$

Then setting $\eta\equiv\inf\{\eta_i,i=1,\dots,m-1\}$, since $\mu_0=\underline{v}$ and $\mu_{m-1}=v$, we obtain v as an implicit function of $\underline{v}$ and the α_i's, for $|\alpha_i|\leq\eta$, $i=1,\dots,m-1$:

$$v=\phi_{m-1}(\alpha_{m-1},\phi_{m-2}(\alpha_{m-2},\cdots,\phi_1(\alpha_1,\underline{v})\cdots)) \tag{3.58}$$

From an application of the implicit function theorem, it results that v may be uniquely written as a function of $(\beta_1,\cdots,\beta_{m-1},\underline{v})$ and using the fact that $\psi_{i,\ell}(0)=0$ and $\psi'_{i,\ell}(0)=0$, it results that the form (3.45) holds true. ■

It remains to establish the following technical result.

Lemma 1 *The determinant*

$$det \begin{cases} i = 1, \dots, m\,;\, i \neq \ell \\ k = 1, \dots, m-1 \end{cases} \left(r_i(\mu_{\ell-1}) \cdot \nabla W_k^\ell(\mu_{\ell-1}) \right)$$

is different from zero.

Proof We consider a null combination of the columns of the $m-1$ square matrix whom element (i,k) is $r_i(\mu_{\ell-1}) \cdot \nabla W_k^\ell(\mu_{\ell-1})$ for $i \neq \ell$ and $k = 1, \dots, m-1$:

$$\sum_{k=1}^{m-1} \omega_k r_i(\mu_{\ell-1}) \cdot \nabla W_k^\ell(\mu_{\ell-1}) = 0\,, i \neq \ell\,, \tag{3.59}$$

that is

$$r_i(\mu_{\ell-1}) \cdot \sum_{k=1}^{m-1} \omega_k \nabla W_k^\ell(\mu_{\ell-1}) = 0\,, i \neq \ell\,. \tag{3.60}$$

Since $r_\ell(\mu_{\ell-1}) \cdot \nabla W_k^\ell(\mu_{\ell-1}) = 0$ we infer that

$$\sum_{k=1}^{m-1} \omega_k \nabla W_k^\ell(\mu_{\ell-1}) = 0. \tag{3.61}$$

But the $m-1$ pseudo-Riemann invariants W_k^ℓ are independent and therefore all the ω_k vanish. This shows that the columns of the aforementioned matrix are independent and therefore its determinant is not zero. ■

Conclusion. We have studied the case $\chi = m-1$, of course the other cases $1 \leq \chi$ are totally similar. As shown by Theorem 4, the two methods lead to close results but for our method, neither k-pseudo Riemann invariants have to be computed, nor a so large system of scalar equations as for the Riemann invariants based method have to be solved.

3.4 Boundary conditions at infinity

Let us consider the case where the physical domain is unbounded, for example the case where the domain is the exterior of a bounded set $\mathcal{S}$. For practical reasons, the computational domain, Ω, must be bounded and this leads to introduce a boundary "far" from $\partial\mathcal{S}$ which is called the boundary at infinity (see Figure 2).

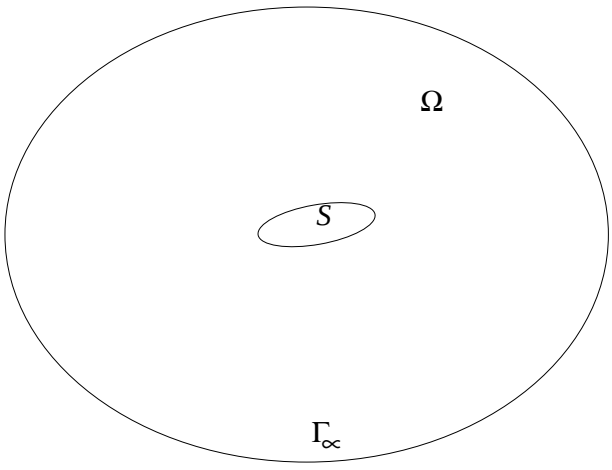

Figure 2: A domain with boundary conditions at infinity.

We want here to discuss the boundary condition treatment on $\Gamma_\infty = \partial\Omega \setminus \partial\mathcal{S}$. More precisely, we consider the case where the physical boundary condition at infinity corresponds to a given state v_∞ such that

$$v \longrightarrow v_\infty \quad \text{as } |x| \longrightarrow \infty\,. \tag{3.62}$$

Let us remark that if the boundary is not sufficiently far from ∂S, then a numerical boundary has to be determined :see Section 3.5.
For a control volume which meets the boundary Γ_∞, we have to find like in Section 3 the normal flux $\Phi(v_K, K, \Gamma_\infty)$ that approximates $\int_{K\cap\Gamma_\infty} F(v(\sigma,t))\cdot\nu_K\, d\sigma$. Consider the eigenvalues $\lambda_k(v_\infty,\nu_K)$ still ordered by increasing order :

$$\lambda_1(v_\infty,\nu_K)\le\lambda_2(v_\infty,\nu_K)\le\ldots\le\lambda_\chi(v_\infty,\nu_K)<0\le\lambda_{\chi+1}(v_\infty,\nu_K)\ldots\le\lambda_m(v_\infty,\nu_K)\,. \tag{3.63}$$

We propose to take :

$$\begin{aligned}\Phi(v_K,K,\Gamma_\infty) &= \sum_{k/\lambda_k(v_\infty,\nu_K)<0} l_k(v_\infty,\nu_K)\cdot(F(v_\infty)\cdot\nu_K)r_k(v_\infty,\nu_K)\\ &+ \sum_{k/\lambda_k(v_\infty,\nu_K)\ge 0} l_k(v_\infty,\nu_K)\cdot(F(v_K)\cdot\nu_K)r_k(v_\infty,\nu_K)\end{aligned} \tag{3.64}$$

This amounts to take $\Phi(v_K,K,\Gamma_\infty)$ solution to the following system :

$$\begin{cases} l_k(v_\infty,\nu_K)\cdot\Phi(v_K,K,\Gamma_\infty)=l_k(v_\infty,\nu_K)\cdot(F(v_\infty)\cdot\nu_K)\,, & k=1,\ldots,\chi\\ \\ l_k(v_\infty,\nu_K)\cdot\Phi(v_K,K,\Gamma_\infty)=l_k(v_\infty,\nu_K)\cdot(F(v_K)\cdot\nu_K)\,, & k=\chi+1,\ldots,m\,.\end{cases} \tag{3.65}$$

Remark 10 *This approach is different from taking :*

$$\Phi(v_K,K,\Gamma_\infty)=F(v_\infty)\cdot\nu_K \tag{3.66}$$

In the case where the code is run in order to find a stationary solution (ie a solution of $divF(v)=0$), it is clear that close to convergence, formulation (3.64) and (3.66) for the normal flux at infinity will give almost the same results. However during transients, formulation (3.64) is better.

3.5 Numerical boundary conditions

3.5.1 More accurate boundary condition at infinity

In Section 3.4, it was assumed that the truncation of the domain does not affect the value on the boundary Γ_∞, we took v_∞ on this boundary as if it was at infinity. This approach turns out to be effective in practice when the boundary Γ_∞ is sufficiently far from the set $\mathcal{S}$. For example, in the case of an airfoil, Γ_∞ has to be at least 50 chords from the profile.
On the other hand, one wishes to have the smallest possible computational domain and therefore one has to modify the previous boundary condition (3.64). A possible method consists in correcting the state v_∞ into an other state say $\tilde v_\infty$. This has to be done via an *ad hoc* procedure depending on the system. Once the state $\tilde v_\infty$ has been chosen, we replace (3.64) by :

$$\begin{aligned}\Phi(v_K,K,\Gamma_\infty) &= \sum_{k/\lambda_k(\tilde v_\infty,\nu_K)\le 0} l_k(\tilde v_\infty,\nu_K)\cdot(F(\tilde v_\infty)\cdot\nu_K)r_k(\tilde v_\infty,\nu_K)\\ &+ \sum_{k/\lambda_k(\tilde v_\infty,\nu_K)>0} l_k(\tilde v_\infty,\nu_K)\cdot(F(v_K)\cdot\nu_K)r_k(\tilde v_\infty,\nu_K)\end{aligned} \tag{3.67}$$

Returning to the case of an airfoil, one obtains $\tilde v_\infty$ by linearizing the Euler equation and by using the so called far field correction (see e.g. Chapter 19.3 of Hirsch [32]).

3.5.2 Cases with symmetries

The numerical boundary condition

$$\Phi(v_K,K,\partial\Omega)=F(v_K)\cdot\nu_K \tag{3.68}$$

can be implemented in at least two cases where numerical boundary reduces the computational domain : the case where the solution is symmetric with respect to an hyperplane and the case where it does not depend on one of the space variables (e.g. one-dimensional computation with a two-dimensional code).

4 An extension to the nonconservative case

Let us now address the case of nonconservative systems. Such systems arise in several contexts of continuous media modelling. Our motivation for studying them stems from two phase fluid mechanics, see Section 7. Here $v = (v_1, \dots, v_m) \in \mathbb{R}^m$ and in general these systems read as :

$$\frac{\partial v}{\partial t} + \nabla \cdot F(v) + \sum_{j=1}^{nd} \tilde{C}_j(v) \frac{\partial v}{\partial x_j} + D(v) \frac{\partial v}{\partial t} = \tilde{S}(v)\,, \tag{4.1}$$

where $\tilde{C}_j(v)$, $D(v)$ are $m \times m$ matrices, $\tilde{S}(v)$ is a m-vector and where

$$\nabla \cdot F(v) = \sum_{j=1}^{nd} \frac{\partial F^j(v)}{\partial x_j}.$$

Functions F^j maps G into $\mathbb{R}^m$, and G is an open subset of $\mathbb{R}^m$ corresponding to the physically admissible states.
We rewrite equation (4.1) as :

$$(\mathbb{1} + D(v)) \frac{\partial v}{\partial t} + \nabla \cdot F(v) + \sum_{j=1}^{nd} \tilde{C}_j(v) \frac{\partial v}{\partial x_j} = \tilde{S}(v)\,, \tag{4.2}$$

and observe that if the matrix $\mathbb{1} + D(v)$ is not invertible, this means that (4.1) is not an evolution equation but rather the coupling of an evolutionary differential equation and of the following nonlinear equation :

$$\nabla \cdot F(v) + \sum_{j=1}^{nd} \tilde{C}_j(v) \frac{\partial v}{\partial x_j} - \tilde{S}(v) \in \text{ Range of } (\mathbb{1} + D(v))\,. \tag{4.3}$$

For example, modelling of incompressible flows lead to such a situation, and (4.3) amounts in this case to the incompressibility condition $div\, u = 0$.
We then assume that the matrix $\mathbb{1} + D(v)$ is invertible. It yields that system (4.1) is equivalent to :

$$\frac{\partial v}{\partial t} + \nabla \cdot F(v) + \sum_{j=1}^{nd} C_j(v) \frac{\partial v}{\partial x_j} = S(v)\,. \tag{4.4}$$

We have kept the flux $F(v)$ unchanged between (4.1) and (4.4). The reason for this is as follows. We assume that (4.1) has been written under this form for physical considerations. The form (4.1) is not unique : indeed one can add to $F(v)$ any function $G(v)$ and (4.1) will remain unchanged for smooth solutions provided we subtract to $\tilde{C}_j(v)$ the Jacobian matrix $\frac{\partial G^j(v)}{\partial v}$. Since there is no uniqueness, the choice for the flux function F must follows from *ad hoc* considerations, like physical ones. This choice will be related to the so-called conservative variables v. Observe that here this denomination is inappropriate since the system is non conservative ! However, once the physical context has been made precise, for example two fluid models like those described in Section 7, the variables which are conserved even in the presence of shocks are known (classically mass, momentum and total energy are conserved). Once these variables v have been chosen, again physical considerations are used to derive their fluxes F. For example in the case of two fluid models, one can ask that these fluxes correspond to that of single fluid model when one of the two phases disappears. Once these fluxes have been chosen, we keep them also in the formulation (4.4). Then we take

$$S(v) = (\mathbb{1} + D(v))^{-1} \tilde{S}(v)$$

and

$$C_j(v) = (\mathbb{1} + D(v))^{-1} \left(\tilde{C}_j(v) + \frac{\partial F^j(v)}{\partial v} \right) - \frac{\partial F^j(v)}{\partial v} \quad j = 1, \dots, nd.$$

Therefore it yields equation (4.4) which is a generalization of (2.1).

4.1 A finite volume discretization

The analog of equation (2.59) is now

$$vol(K) \frac{dv_K}{dt} + F^{\nu}_{\partial K} + \int_K \sum_{j=1}^{nd} C_j(v) \frac{\partial v}{\partial x_j}\, dx = \int_K S(v)\, dx\,. \tag{4.5}$$

Hence the time evolution of $v_K(t)$ is governed by 3 terms :

$$F^{\nu}_{\partial K}(t) = \int_{\partial K} F(v(\sigma,t)) \cdot \nu(\sigma)\, d\sigma, \tag{4.6}$$

$$\int_K \sum_{j=1}^{nd} C_j(v) \frac{\partial v}{\partial x_j}\, dx\,, \tag{4.7}$$

$$\int_K S(v)\, dx\,. \tag{4.8}$$

The first one, (4.6), is the normal flux on the boundary of K, and in order to approximate it, we use again a formula that reads as (2.63) : more details are given in Section 4.1.1 The second one is a non conservative product which will be dealt in Section 4.1.2. Finally the third one is a source term *i.e.* it contains no derivative. For the later, a natural discretization reads as :

$$\int_K S(v)\, dx \approx vol(K) S(v_K)\,. \tag{4.9}$$

4.1.1 Approximation of the conservative term

As in Section 2.3.1, $F^{\nu}_{\partial K}$ is decomposed in a sum :

$$F^{\nu}_{\partial K} = \sum_{L \in \mathcal{N}(K)} F_{K,L}\,, \quad \text{with} \quad F_{K,L} = \int_{K\cap L} F(v(\sigma,t)) \cdot \nu_{K,L}\, d\sigma\,. \tag{4.10}$$

and the integral on $K \cap L$ is approximated by a numerical flux $\phi(v_K, v_L, K, L)$. Here this later is obtained from "VFFC" method by formula (2.66) when we take :

$$U(v, w; K, L) = \text{sgn}(\tilde{A}_\nu(\mu(v, w; K, L)))\,, \tag{4.11}$$

where $\mu(v, w; K, L)$ is a mean between v and w depending on the geometry, e.g. (2.68). Let us now defined the matrix $\tilde{A}_\nu(v)$.
In order to find an expression for the function $\Phi(v, w; K, L)$, we write equation (4.4) in coordinates which are adapted to the edge $K \cap L$. For fixed adjacent volumes K and L, we denote by $(\nu, \tau_1, \ldots, \tau_{nd-1})$ these coordinates : ν is the unit normal to the interface $K \cap L$ oriented from K to L and $\tau_1, \ldots, \tau_{nd-1}$ is an orthonormal basis of the hyperplane orthogonal to ν. By a change of independent variables and with natural notations ($F_\nu(v) = F(v) \cdot \nu$, $C_\nu(v) = C(v) \cdot \nu$, $F_\tau(v) = \sum_{i=1}^{nd-1} F(v) \cdot \tau_i$), we can rewrite equation (4.4) as :

$$\frac{\partial v}{\partial t} + \frac{\partial F_\nu(v)}{\partial \nu} + C_\nu(v) \frac{\partial v}{\partial \nu} + \nabla_\tau \cdot F_\tau(v) + \sum_{i=1}^{nd-1} C_{\tau_i}(v) \frac{\partial v}{\partial \tau_i} = S(v)\,. \tag{4.12}$$

Next, introducing the matrix

$$E_\nu(v) \equiv C_\nu(v) J_\nu(v)^{-1} \quad \text{where} \quad J_\nu(v) \equiv \frac{\partial F_\nu(v)}{\partial v}, \tag{4.13}$$

we observe that (4.12) can be written as :

$$\frac{\partial v}{\partial t} + (\mathbb{1} + E_\nu(v)) \frac{\partial F_\nu(v)}{\partial \nu} + \nabla_\tau \cdot F_\tau(v) + \sum_{i=1}^{nd-1} C_{\tau_i}(v) \frac{\partial v}{\partial \tau_i} = S(v)\,. \tag{4.14}$$

Hence the unknown $F_\nu(v)$, that we want to approximate, satisfies at the continuous level the following evolution equation :

$$\begin{aligned} &\frac{\partial F_\nu(v)}{\partial t} + J_\nu(v)(\mathbb{1} + E_\nu(v)) \frac{\partial F_\nu(v)}{\partial \nu} \\ &\qquad + J_\nu(v) \nabla_\tau \cdot F_\tau(v) + \sum_{i=1}^{nd-1} J_\nu(v) C_{\tau_i}(v) \frac{\partial v}{\partial \tau_i} = J_\nu(v) S(v)\,. \end{aligned} \tag{4.15}$$

Since our goal is to approximate the integral of $F_\nu(v)$ on $K \cap L$, we first linearize equation (4.15) on the edge $K \cap L$ by using a mean value for v on $K \cap L$, let say $\mu = \mu(v_K, v_L, K, L)$ and we obtain :

$$\begin{aligned} &\frac{\partial F_\nu(v)}{\partial t} + J_\nu(\mu)(\mathbb{1} + E_\nu(\mu)) \frac{\partial F_\nu(v)}{\partial \nu} \\ &\qquad + J_\nu(\mu) \nabla_\tau \cdot F_\tau(v) + \sum_{i=1}^{nd-1} J_\nu(\mu) C_{\tau_i}(\mu) \frac{\partial v}{\partial \tau_i} = J_\nu(\mu) S(\mu)\,. \end{aligned} \tag{4.16}$$

Now integrating this last equation on $K \cap L$ leads to the approximate convection equation for the flux $F_{K,L}$, defined in (2.62) :

$$\frac{\partial F_{K,L}}{\partial t} + J_\nu(\mu)(\mathbb{1} + E_\nu(\mu))\frac{\partial F_{K,L}}{\partial \nu} = area(K \cap L) J_\nu(\mu) S(\mu)\,. \tag{4.17}$$

Indeed $d\sigma = d\tau_1 \dots d\tau_{nd-1}$ and the differential terms in the second line of equation (4.16) are exact derivatives with respect to the τ_i, hence they only contribute, after integration on $K \cap L$, to terms at the boundary of $K \cap L$. Since the present finite volume discretization method understands that exchanges between the volumes occur through edges, terms which involves edges boundaries are neglected. Let us define the matrix :

$$\tilde{A}_\nu(v) \equiv J_\nu(v)(\mathbb{1} + E_\nu(v)) = J_\nu(v) + J_\nu(v) C_\nu(v) J_\nu(v)^{-1}\,, \tag{4.18}$$

we see from equation (4.17) that the normal flux is convected, in first approximation, by the matrix $\tilde{A}_\nu(\mu)$. This fact leads to the choice (4.11).

4.1.2 Discretization of the nonconservative product

Our goal now is to approximate the integral (4.7) which involves the non conservative products. Let us note $\nu = \nu_{K,L}$ and $\mu = \mu(v_K, v_L, K, L)$. We write successively :

$$\begin{aligned}
\int_K \sum_{j=1}^{nd} C_j(v)\frac{\partial v}{\partial x_j}\,dx &= \int_K \sum_{j=1}^{nd} C_j(v)\frac{\partial (v - v_K)}{\partial x_j}\,dx\,,\\
&\approx \sum_{j=1}^{nd} C_j(v_K)\int_K \frac{\partial (v - v_K)}{\partial x_j}\,dx\,,\\
&\approx \sum_{j=1}^{nd} C_j(v_K)\int_{\partial K} \nu^j (v(\sigma,t) - v_K)\,d\sigma\,,
\end{aligned}$$

and we decompose the last integral into a sum :

$$\int_K \sum_{j=1}^{nd} C_j(v)\frac{\partial v}{\partial x_j}\,dx \approx \sum_{L\in\mathcal{N}(K)} \left(\sum_{j=1}^{nd} C_j(v_K)\cdot\nu^j\right)\int_{K\cap L} (v(\sigma,t) - v_K)\,d\sigma\,. \tag{4.19}$$

Then, on $K \cap L$, we approximate :

$$v(\sigma,t) - v_K \approx J_\nu(\mu)^{-1}\left(F(v(\sigma,t))\cdot\nu - F(v_K)\cdot\nu\right) \tag{4.20}$$

so that, with $E_{K,L} \equiv C_\nu(v_K) J_\nu(\mu)^{-1}$,

$$\int_K \sum_{j=1}^{nd} C_j(v)\frac{\partial v}{\partial x_j}\,dx \approx \sum_{L\in\mathcal{N}(K)} E_{K,L}\int_{K\cap L} \left(F(v(\sigma,t))\cdot\nu - F(v_K)\cdot\nu\right)\,d\sigma\,. \tag{4.21}$$

Finally using the approximation of the conservative term described in the previous section, we arrive to the following formula :

$$\int_K \sum_{j=1}^{nd} C_j(v)\frac{\partial v}{\partial x_j}\,dx \approx \sum_{L\in\mathcal{N}(K)} area(K\cap L)\, E_{K,L}(\Phi(v_K, v_L; K, L) - F(v_K)\cdot\nu)\,. \tag{4.22}$$

4.1.3 Space discretization : summary

Now we can discretize (4.5) as follows :

$$\begin{aligned}
&vol(K)\frac{dv_K}{dt} \\
&\quad + \sum_{L\in\mathcal{N}(K)} area(K\cap L)\,(\mathbb{1} + E_{K,L})(\Phi(v_K, v_L; K, L) - F(v_K)\cdot\nu_{K,L})\\
&\quad = vol(K) S(v_K)\,.
\end{aligned} \tag{4.23}$$

with $E_{K,L} = C_{\nu_{K,L}}(v_K) J_{\nu_{K,L}}(\mu(v_K, v_L, K, L))^{-1}$

4.2 Discretization of the boundary conditions

Let us study the extension of our method and its comparison with the Riemann invariant based technique.

4.2.1 Extension of our method

Exactly as for the conservative case, we have to find the numerical flux $\Phi(v_K, K, \partial\Omega)$ that approximates $\int_{K\cap\partial\Omega} F(v(\sigma,t))\cdot\nu_K\, d\sigma$. Here the linearization around the state $\underline{v}$ still reads (3.3) but this time the advection matrix $\underline{A}_{\nu_K}$ is :

$$\underline{A}_{\nu_K} \equiv J_{\nu_K}(\underline{v}) + C_\nu(\underline{v})\,. \tag{4.24}$$

As in the conservative case, there are two different situations. The first one (the non characteristic case) refers to the case where the matrix $\underline{A}_{\nu_K}$ is invertible, while the second one refers to the complementary case.

The non characteristic case The eigenvalues $\lambda_k(\underline{v},\nu_K)$ of $\underline{A}_{\nu_K}$ are :

$$\lambda_1(\underline{v},\nu_K) \le \lambda_2(\underline{v},\nu_K) \le \ldots \le \lambda_\chi(\underline{v},\nu_K) < 0 < \lambda_{\chi+1}(\underline{v},\nu_K)\ldots \le \lambda_m(\underline{v},\nu_K)\,.$$

and like in the conservative case, we have to discuss according to the value of χ. There is no change for the cases $\chi = 0$ and $\chi = m$. In the case $1 \le \chi \le m-1$, the discussion is exactly that of Section 3.1.3. There is only one difference, we take for $h_k(v)$:

$$h_k(v) \equiv \tilde{l}_k(\underline{v},\nu_K)\cdot(F(v)\cdot\nu_K) - \tilde{l}_k(\underline{v},\nu_K)\cdot(F(v_K)\cdot\nu_K)\,,\quad k=\chi+1,\ldots,m\,, \tag{4.25}$$

where this time $\tilde{l}_k(v,\nu)$ denotes a left eigenvector of $\tilde{A}_\nu(v)$.

The characteristic case Here we label the eigenvalues of the matrix $\underline{A}_{\nu_K}$ as follows :

$$\begin{aligned}&\lambda_1(\underline{v},\nu_K) \le \lambda_2(\underline{v},\nu_K) \le \ldots \le \lambda_\chi(\underline{v},\nu_K) < 0 \lambda_{\chi+n_0+1}(\underline{v},\nu_K)\ldots \le \lambda_m(\underline{v},\nu_K)\,,\\ &\lambda_{\chi+1}(\underline{v},\nu_K) = \ldots = \lambda_{\chi+n_0}(\underline{v},\nu_K) = 0\,.\end{aligned}$$

Here $n_0 \ge 1$ denotes the dimension of the kernel of $\underline{A}_{\nu_K}$. There is no change for the case $\chi = 0$ and the case $\chi = m - n_0$. In the case $1 \le \chi \le m-n_0-1$, the discussion is exactly that of Section 3.2.3. There is only one difference, we take again for the $h_k(v)$'s the functions given by (4.25).

4.2.2 The incomplete Riemann invariant technique

Let us sum up the argument we developed for the conservative case in Section 3.3.3 for the case $\chi = m-1$ and let us consider at each interface the normal equation that reads :

$$\frac{\partial v}{\partial t} + \frac{\partial f(v)}{\partial x} + C(v)\frac{\partial v}{\partial x} = S(v)\,. \tag{4.26}$$

We denote by $J(v)$ the Jacobian matrix $\frac{\partial f(v)}{\partial v}$ and by $A(v) = J(v)+C(v)$ and $\tilde{A}(v) = J(v)+J(v)C(v)J^{-1}(v)$. In the non-conservative case, the "incomplete" Riemann problem method should consist in finding a state v satisfying the $m-1$ physical boundary conditions :

$$g_k(v) = 0\,,\quad k = 1,\ldots,m-1\,, \tag{4.27}$$

and $m-2$ intermediate states $\mu_1,\ldots,\mu_{m-2}$ such that $\underline{v}$ is connected to μ_1 through a 1-simple wave, μ_1 is connected to μ_2 through a 2-simple wave, ..., μ_{m-2} is connected to v through a $(m-1)$-simple wave *i.e.* such that with $\mu_0 \equiv v$ and $\mu_{m-1} \equiv \underline{v}$:

$$W_k^\ell(\mu_{\ell-1}) = W_k^\ell(\mu_\ell)\,, k = 1,\ldots,m-1\,, \ell = 1,\ldots,m-1\,. \tag{4.28}$$

where $\{W_k^\ell\}_{k=1}^{m-1}$ are the $m-1$ independent ℓ-pseudo Riemann invariants, that therefore satisfy :

$$r_\ell(v)\cdot\nabla W_k^\ell(v) = 0\,. \tag{4.29}$$

Here $r_\ell(v)$ denotes a right eigenvector of $A(v)$. The method we propose replaces the system (4.28) by the single equation :

$$\tilde{l}_m(\underline{v})\cdot f(v) = \tilde{l}_m(\underline{v})\cdot f(\underline{v})\,. \tag{4.30}$$

where $\tilde{l}_k(v)$ denotes a left eigenvector of $\tilde{A}(v)$.

Theorem 4 holds true with

$$E_\epsilon = \left\{ v \in G \,/\, ||v - \underline{v}|| \leq \epsilon \text{ and } \tilde{l}_m(\underline{v}) \cdot f(v) = \tilde{l}_m(\underline{v}) \cdot f(\underline{v}) \right\} , \tag{4.31}$$

since if we take $v = \underline{v} + \sum_{i=1}^m \alpha_i r_i(\underline{v})$ and if we write the Taylor expansion :

$$f(v) = f(\underline{v}) + J(\underline{v})(v - \underline{v}) + \int_0^1 (1-\theta) \frac{\partial J}{\partial v}(\theta v + (1-\theta)\underline{v})(v - \underline{v}, v - \underline{v}) d\theta \,,$$

and then by taking the product scalar with $\tilde{l}_m(\underline{v})$, we get

$$\tilde{l}_m(\underline{v}) \cdot f(v) = \tilde{l}_m(\underline{v}) \cdot f(\underline{v}) + \alpha_m + \tilde{l}_m(\underline{v}) \cdot \int_0^1 (1-\theta) \frac{\partial J}{\partial v}(\theta v + (1-\theta)\underline{v})(v - \underline{v}, v - \underline{v}) d\theta \,.$$

since $\tilde{l}(\underline{v}) = {}^t J^{-1}(\underline{v}) l(\underline{v})$. We now see that (4.30) reads as :

$$\alpha_m + \tilde{g}(\alpha_1, \ldots, \alpha_m) = 0 \,. \tag{4.32}$$

5 Wave equations

In this Section, we discuss how the present boundary conditions treatment applies in the cases of Example 1 namely the second order linear wave and Example 2 namely Maxwell's equations. These two situations correspond to a linear hyperbolic system of conservation laws where we observe that, for $nd \geq 2$, we always have a zero eigenvalue. It means that the boundary is always characteristic and the general Theorem 1 cannot be applied. Nevertheless as it has been noticed in Section 3.2, we are going to be able to find in each case a unique normal flux at the boundary by solving directly the system of equations (3.9) and (3.13). From physical point of view, the zero eigenvalues corresponds to crawling (or surface) waves that can live on the boundary.
Let us recall that K is a control volume that meets $\partial\Omega$, ν_K is the unit external normal and $\underline{v}$ denotes v_K

5.1 The multidimensional wave equation

For this problem, the number of unknowns is equal to $nd + 1$ and the second order equation reads as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \triangle u = 0 \,. \tag{5.1}$$

According to the definition of the eigenvalues (2.26), $nd - 1$ eigenvalues are equal to 0 ; the boundary is characteristic and the hypothesis of Theorem 1 does not hold. But since one eigenvalue, namely $\lambda_{nd+1}(\nu_K) = c$ is positive, we know that one information comes from inside Ω and in order to determine the normal flux $F(v) \cdot \nu$ at the boundary, if we follow our strategy, we have to impose (3.13) which reads :

$$l_{nd+1}(\nu_K) \cdot (F(v) \cdot \nu_K) = l_{nd+1}(\nu_K) \cdot (F(v_K) \cdot \nu_K) \,, \tag{5.2}$$

that is, using (2.5) and (2.27),

$$\frac{c}{\sqrt{2}}(1, \nu_K) \cdot (v_\sharp \cdot \nu_K, u\nu_K) = \frac{c}{\sqrt{2}}(1, \nu_K) \cdot (\underline{v}_\sharp \cdot \nu_K, \underline{u}\nu_K), \tag{5.3}$$

ie

$$v_\sharp \cdot \nu_K + u = \underline{v}_\sharp \cdot \nu_K + \underline{u} \,. \tag{5.4}$$

Remark 11 *This condition corresponds to the conservation of a Riemann invariant across the boundary for the first order system (2.3)-(2.4), see Section 3.3.*

Remark 12 *Here, it is much more simple to try to find the normal flux at the boundary that is* $(v_\sharp \cdot \nu_K, u\nu_K)$ *than the variable* $(v_\sharp, u)$.

Remark 13 *Let us note that for all* v *and* v_K*, we have*

$$l_k(\nu_K) \cdot (F(v) \cdot \nu_K - F(v_K) \cdot \nu_K) = 0 \,, k = 2, \ldots, nd. \tag{5.5}$$

From these relations and from (5.2), we can write

$$\begin{aligned} F(v) \cdot \nu_K &= \sum_{k=1}^{nd+1} l_k(\nu_K) \cdot (F(v) \cdot \nu_K) r_k(\nu_K) \,, \\ &= F(\underline{v}) \cdot \nu_K + l_1(\nu_K) \cdot (F(v) \cdot \nu_K - F(\underline{v}) \cdot \nu_K) r_1(\nu_K), \end{aligned}$$

and $F(v)\cdot\nu_K$ can be seen as a perturbation of $F(\underline{v})\cdot\nu_K$

$$F(v)\cdot\nu_K \quad = F(\underline{v})\cdot\nu_K + \varepsilon\, r_1(\nu_K)\,,$$

with

$$\varepsilon = \frac{c}{\sqrt{2}}(v_\sharp\cdot\nu_K - u - \underline{v}_\sharp\cdot\nu_K + \underline{u}) \tag{5.6}$$

and from (5.4)

$$\varepsilon \equiv c\sqrt{2}(\underline{u}-u) \equiv c\sqrt{2}(v_\sharp\cdot\nu_K - \underline{v}_\sharp\cdot\nu_K).$$

It remains to find one equation linearly independent from equation (5.4) : let say $g_1(v)=0$. A great range of boundary conditions can be considered, let us focus on the most usual ones that is Dirichlet, Neumann and Robin conditions.

5.1.1 The Dirichlet case

Here this condition $g_1(v)=0$ reads

$$u = u_{given}\,,\ \text{on}\ \Gamma_D\,, \tag{5.7}$$

where Γ_D is the part of $\partial\Omega$ on which the function u is given. Equation (5.4) leads then to

$$v_\sharp\cdot\nu_K = \underline{v}_\sharp\cdot\nu_K + \underline{u} - u_{given}\,, \tag{5.8}$$

so that $\varepsilon = c\sqrt{2}(\underline{u}-u_{given})$ and the normal flux at the boundary is explicitly determined by :

$$F(v)\cdot\nu_K = F(\underline{v})\cdot\nu_K + c(\underline{u}-u_{given})(1,-\nu_K)\,. \tag{5.9}$$

5.1.2 The Neumann case

Here this condition $g_1(v)=0$ reads

$$\frac{\partial u}{\partial\nu} \equiv \nu_K\cdot\nabla u = 0\,,\ \text{on}\ \Gamma_N\,, \tag{5.10}$$

where Γ_N is the part of $\partial\Omega$ on which the Neumann condition is imposed. We observe that according to (2.4), this condition is equivalent to

$$v_\sharp\cdot\nu_K = v^0_\sharp\cdot\nu_K\,. \tag{5.11}$$

But according to (5.4), it gives again ε explicitly : $\varepsilon = -c\sqrt{2}(\underline{v}_\sharp\cdot\nu_K - v^0_\sharp\cdot\nu_K)$. Hence finally we have the following normal flux at the boundary

$$F(v)\cdot\nu_K = F(\underline{v})\cdot\nu_K - c(\underline{v}_\sharp\cdot\nu_K - v^0_\sharp\cdot\nu_K)(1,-\nu_K)\,. \tag{5.12}$$

5.1.3 The Robin case

The complementary condition reads

$$\frac{\partial u}{\partial\nu} + \gamma u = 0\,,\ \text{on}\ \Gamma_R\,, \tag{5.13}$$

where Γ_R is the part of $\partial\Omega$ on which the Robin condition is imposed and γ is a given parameter. Following the same lines as in the previous cases, we arrive to the following differential equation :

$$\nu_K\cdot\frac{\partial v_\sharp}{\partial t} = \gamma c u\,. \tag{5.14}$$

Using equation (5.4), we deduce that

$$\frac{\partial}{\partial t}\left(e^{-\gamma c t}v_\sharp\cdot\nu_K\right) = \gamma c(\underline{u}+\underline{v}_\sharp\cdot\nu_K)e^{-\gamma c t}\,. \tag{5.15}$$

Integration of this differential equation leads then to

$$v_\sharp\cdot\nu_K = v^0_\sharp\cdot\nu_K - \gamma c\int_0^t(\underline{u}+\underline{v}_\sharp\cdot\nu_K)e^{\gamma c(t-s)}ds\,. \tag{5.16}$$

But also, (5.15) can be approximatively integrated between t_{n-1} and t_n to lead to (assuming $\underline{u}$ and $\underline{v}_\sharp\cdot\nu_K$ independent of time, for instance $\underline{u}=u_K^{n-1}$ and $\underline{v}_\sharp\cdot\nu_K = v^{n-1}_{\sharp,K}\cdot\nu_K$)

$$v^n_\sharp\cdot\nu_K = v^{n-1}_\sharp\cdot\nu_K + (\underline{u}+\underline{v}_\sharp\cdot\nu_K)(1-e^{\gamma c(t_n-t_{n-1})})\,. \tag{5.17}$$

But according to (5.6) this gives ε and hence finally we have the following normal flux at the boundary

$$F(v)\cdot\nu_K = F(\underline{v})\cdot\nu_K - c\left(\underline{v}_\sharp\cdot\nu_K e^{\gamma c(t_n-t_{n-1})} - v^{n-1}_\sharp\cdot\nu_K - \underline{u}(1-e^{\gamma c(t_n-t_{n-1})})\right)(1,-\nu_K)\,.$$

5.2 Maxwell's equations

Recall that this system of equations reads as ($m = 2nd$)

$$\frac{\partial D}{\partial t} - curl H = 0\,, \tag{5.18}$$

$$\frac{\partial B}{\partial t} + curl E = 0\,, \tag{5.19}$$

with constitutive equations $D = \epsilon E$ and $B = \mu H$.
According to (2.29), and with the notations of Section 3, we are in the case $\chi = nd - 1$. Therefore we need $nd - 1$ scalar information coming from outside of Ω. The $nd + 1$ supplementary conditions (3.13) amounts to say that there exist $\varepsilon_j \in \mathbb{R}$ for $j = 1, \ldots, nd - 1$ such that

$$F(v) \cdot \nu_K = F(\underline{v}) \cdot \nu_K + \sum_{j=1}^{nd-1} \varepsilon_j \, r_j(\nu_K)\,, \tag{5.20}$$

that is with (2.30)

$$F(v) \cdot \nu_K = F(\underline{v}) \cdot \nu_K + (\Pi, -\frac{\nu_K \wedge \Pi}{c\epsilon})\,, \tag{5.21}$$

where Π is an arbitrary vector in $\mathbb{R}^{nd}$ orthogonal to ν_K. Hence in order to find the normal flux on the boundary, we have to find the vector Π which depends on $nd - 1$ independent variables and will be determined by the same number of physical boundary conditions. There are mainly 3 types of such conditions that we study now.

5.2.1 Wall conditions

Also known as obstacle conditions, they can be (at least) of two nature : the case of a perfectly conducting surface and the case of a perfectly reflecting surface.

Perfectly conducting surface. In this case, the tangent part of the electric field E is given

$$\nu_K \wedge D = \nu_K \wedge D_{given}\,. \tag{5.22}$$

Using (5.21) we find Π explicitly and then we have the following normal flux at the boundary

$$\begin{aligned} F(v) \cdot \nu_K = \quad & F(\underline{v}) \cdot \nu_K \\ & + \left(c\nu_K \wedge (\nu_K \wedge (\underline{D} - D_{given})), \frac{\nu_K \wedge (D_{given} - \underline{D})}{\epsilon} \right)\,. \end{aligned} \tag{5.23}$$

Actually, another way of considering a perfectly conducting surface might consist in saying that the whole field (D, B) is given outside of Ω and in using the characteristics that enter the domain in order to find the normal flux at the boundary. These characteristics correspond to the negative eigenvalues in (2.29) and therefore, consistently with (3.13), we impose :

$$l_k(\nu_K) \cdot (F(v) \cdot \nu_K) = l_k(\nu_K) \cdot (F(v_{given}) \cdot \nu_K)\,, \quad k = 1, \ldots, nd - 1\,. \tag{5.24}$$

This gives then automatically the normal flux through the formula

$$\begin{aligned} F(v) \cdot \nu_K \quad = \quad & \sum_{k=1}^{nd-1} l_k(\nu_K) \cdot (F(v_{given}) \cdot \nu_K) r_k(\nu_K) \\ + \quad & \sum_{k=nd}^{2nd} l_k(\nu_K) \cdot (F(\underline{v}) \cdot \nu_K) r_k(\nu_K)\,. \end{aligned} \tag{5.25}$$

In general, formulas (5.23) and (5.25) do not lead to the same normal flux on the boundary since (5.23) does not involve B_{given} while (5.25) does. Depending on the physical situation one should decide between the two numerical boundary conditions.

Perfectly reflecting surface. On a perfectly reflecting surface, one can either impose that the tangent part of the electric field E vanishes

$$\nu_K \wedge D = 0\,, \tag{5.26}$$

or that the whole field (D, B) vanishes in the bulk outside of Ω. This last condition corresponds to formula (5.24) with $v_{given} = 0$ and according to (5.25) leads to

$$F(v)\cdot\nu_K = \sum_{k=nd}^{2nd} l_k(\nu_K)\cdot(F(\underline{v})\cdot\nu_K) r_k(\nu_K)\,. \tag{5.27}$$

On the other hand, according to (5.23), condition (5.26) leads to the following normal flux

$$F(v)\cdot\nu_K = F(\underline{v})\cdot\nu_K + \left(c\nu_K\wedge(\nu_K\wedge\underline{D}), -\frac{\nu_K\wedge\underline{D}}{\epsilon}\right)\,. \tag{5.28}$$

5.2.2 Absorbing boundary conditions

The first order Silver-Müller absorbing condition reads as :

$$\nu_K\wedge D = -\sqrt{\frac{\epsilon}{\mu}}\nu_K\wedge(\nu_K\wedge B)\,. \tag{5.29}$$

Again this allows ones to find Π in (5.21) and leads to the following normal flux at the boundary

$$\begin{aligned} &F(v)\cdot\nu_K = F(\underline{v})\cdot\nu_K + \\ &\frac{c}{2}\left(\sqrt{\frac{\epsilon}{\mu}}\nu_K\wedge\underline{B} - \nu_K\wedge(\nu_K\wedge\underline{D}), \frac{1}{c\epsilon}\left(\nu_K\wedge\underline{D} + \sqrt{\frac{\epsilon}{\mu}}\nu_K\wedge(\nu_K\wedge\underline{B})\right)\right)\,. \end{aligned} \tag{5.30}$$

6 The Euler equations for inviscid fluids

In this Section, we are going to discuss in details the boundary conditions associated with the multidimensional Euler equations for inviscid fluids (2.17) to (2.19). Let us recall that the number of unknowns m is equal to $nd+2$, K is a control volume that meets $\partial\Omega$, ν_K is the unit external normal and $\underline{v}$ denotes v_K. Let us notice that for ν a given unit vector, we have the formula (see (2.20)) :

$$F(v)\cdot\nu = (\rho(u\cdot\nu), \rho(u\cdot\nu)u + p\nu, \rho H(u\cdot\nu))\,. \tag{6.1}$$

6.1 Wall conditions

On a wall, the fluid slips, it means that its normal velocity $u\cdot\nu_K$ is equal to 0. Hence the normal flux $F(v)\cdot\nu_K$ is equal to

$$F(v)\cdot\nu_K = (0, p\nu_K, 0).$$

Therefore, to determine this flux, we only have to find the unknown p.

Remark 14 *Here it is much more simple and much more physically relevant to try to find the normal flux on the boundary rather than the variable $v = (\rho, \rho u, \rho E)$.*

■

According to the value of the eigenvalues (2.50), only one eigenvalue, namely $\lambda_{nd+2}(v,\nu_K) = c$, is positive. Only one information comes from inside Ω and if we follow the strategy we have described in Section 3.2, we have to impose (3.23) which reads :

$$l_{nd+2}(\underline{v},\nu_K)\cdot(0,p\nu_K,0) = l_{nd+2}(\underline{v},\nu_K)\cdot(F(\underline{v})\cdot\nu_K)\,. \tag{6.2}$$

Hence pressure p is automatically given by the explicit formula

$$p = \frac{l_{nd+2}(\underline{v},\nu_K)\cdot(F(\underline{v})\cdot\nu_K)}{l_{nd+2}(\underline{v},\nu_K)\cdot(0,\nu_K,0)}\,, \tag{6.3}$$

that is (where underlined quantities correspond to $\underline{v}$) :

$$p = \underline{p} + \frac{\underline{\rho c}^2(\underline{u}\cdot\nu_K)}{\underline{c} - \underline{k}(\underline{u}\cdot\nu_K)}\,. \tag{6.4}$$

In other words we take

$$\Phi(v_K, K, \partial\Omega) = (0, p\nu_K, 0)\,, \tag{6.5}$$

where p is given by formula (6.4) with $\underline{v} = v_K$.

Remark 15 *For a polytropic gas, for which the equation of state is $p = (\gamma - 1)\rho e$ where $\gamma > 1$ is a given constant, we have $k = \gamma - 1$ and $c^2 = \frac{\gamma p}{\rho}$. Then formula (6.4) reads as*

$$p = \underline{p}\left(1 + \frac{\gamma \underline{u}\cdot\nu_K}{\underline{c} - (\gamma-1)\underline{u}\cdot\nu_K}\right) . \tag{6.6}$$

6.2 Fluid boundary conditions

Let us now turn to the case where on a part of the boundary the fluid goes inside or outside the domain, that is $u\cdot\nu_K \neq 0$. We are going to discuss according to the sign of this normal velocity.

6.2.1 The case of an outlet : $u\cdot\nu_K > 0$

The supersonic case. This is the case where $u\cdot\nu_K > c$. Here $\chi = 0$, so we have all the information that comes from inside of Ω and according to Section 3.1.1 we take :

$$\Phi(v_K, K, \partial\Omega) = F(v_K)\cdot\nu_K\,. \tag{6.7}$$

The subsonic case. This is the case where $0 < u\cdot\nu_K < c$. We have $nd+1$ information that come from inside of Ω and $\chi = 1$. Therefore, we have to prescribe one boundary condition[5]. Let us write :

$$g_1(v) = 0\,, \tag{6.8}$$

the given relation at the boundary. Condition (3.15) reads here

$$\Delta \equiv \frac{\partial g_1}{\partial v_1}(\underline{v}) + \sum_{i=1}^{nd}(\underline{u}_i - \underline{c}\nu_K^i)\frac{\partial g_1}{\partial v_{i+1}}(\underline{v}) + (\underline{H} - \underline{u}\cdot\nu_K\underline{c})\frac{\partial g_1}{\partial v_{nd+2}}(\underline{v}) \neq 0\,. \tag{6.9}$$

The given pressure case. We consider the usual case where the given physical boundary condition at a subsonic outlet consist in imposing a given pressure. Let us denote p_{OUT} this pressure, then the condition (6.8) reads $g_1(v) \equiv p - p_{OUT} = 0$. In order to check condition (6.9) in this case, we are going to compute the gradient $\nabla_v p_{|v=\underline{v}}$. Here we have

$$\rho = v_1\,, \tag{6.10}$$

$$e = \frac{v_{nd+2}}{v_1} - \frac{1}{2}\frac{\sum_{i=2}^{nd+1} v_i^2}{v_1^2}\,, \tag{6.11}$$

so that we find that

$$\frac{\partial p}{\partial v_1} = \left(\frac{\partial p}{\partial \rho}\right)_e - \frac{e - \frac{1}{2}u^2}{\rho}\left(\frac{\partial p}{\partial e}\right)_\rho\,, \tag{6.12}$$

$$\frac{\partial p}{\partial v_i} = -\frac{u_{i-1}}{\rho}\left(\frac{\partial p}{\partial e}\right)_\rho, i = 2,\ldots,nd+1\,, \tag{6.13}$$

$$\frac{\partial p}{\partial v_{nd+2}} = \frac{1}{\rho}\left(\frac{\partial p}{\partial e}\right)_\rho . \tag{6.14}$$

If we substitute these expressions in (6.9), we obtain

$$\Delta = \left(\frac{\partial p}{\partial \rho}\right)_{e\big|_{v=\underline{v}}} + \frac{\underline{p}}{\underline{\rho}^2}\left(\frac{\partial p}{\partial e}\right)_{\rho\big|_{v=\underline{v}}} . \tag{6.15}$$

On the other hand, if we write that

$$Tds = de - \frac{p}{\rho^2}d\rho\,, \tag{6.16}$$

we obtain that (see (2.49)) :

$$dp = c^2 d\rho + k\rho T ds = \left[\left(\frac{\partial p}{\partial \rho}\right)_e + \frac{p}{\rho^2}\left(\frac{\partial p}{\partial e}\right)_\rho\right]d\rho + T\left(\frac{\partial p}{\partial e}\right)_\rho ds\,, \tag{6.17}$$

so that we simply have $\Delta = \underline{c}^2$, a positive number. Hence we have proved the following result :

Proposition 2 *In the case of a subsonic outlet, we can only prescribe one condition $g_1(v) = 0$. Moreover this condition is suitable if and only if (6.9) is satisfied. In particular one can impose the pressure p.*

[5] In most cases it is the pressure that is given at a subsonic outlet, see hereafter.

6.2.2 The case of an inlet : $u \cdot \nu_K < 0$

The supersonic case. This is the case where $u \cdot \nu_K < -c$. Then $\chi = nd + 2$ so that we have all the information that comes from outside of Ω and according to Section 3.1.2 we have to prescribe the flux on the boundary :

$$\Phi(v_K, K, \partial\Omega) = \Phi_{given} \,. \tag{6.18}$$

The subsonic case. This is the case where $-c < u \cdot \nu_K < 0$. Here $\chi = nd + 1$, we need one information that comes from inside of Ω and we have to prescribe $nd + 1$ boundary conditions.
In general one prescribes the direction of the flow on the boundary *i.e.* $u = \mu\alpha$ where α is a unit vector which makes an obtuse angle with ν_K with $\alpha \cdot \nu_K < 0$ and where μ is a positive number such that $-\alpha \cdot \nu_K \, \mu < c$. Giving α amounts to give $nd - 1$ conditions and therefore it remains to prescribe 2 boundary conditions.

Let us first discuss the important case where the flow is normal to the boundary *i.e.* $\alpha = -\nu_K$. In this case, $0 < \mu < c$ and

$$F(v) \cdot \nu_K = -(\rho\mu, (\rho\mu^2 + p)\nu_K, \rho\mu H) \,. \tag{6.19}$$

Remark 16 *We recognize in this formula the flux that occurs in the one dimensional case, see (10.3).*

In order to find this normal flux, according to (3.13), we first take

$$l_{nd+2}(\underline{v}, \nu_K) \cdot (F(v) \cdot \nu_K) = l_{nd+2}(\underline{v}, \nu_K) \cdot (F(\underline{v}) \cdot \nu_K) \,, \tag{6.20}$$

since one information is coming from inside Ω. Actually we propose in this case to slightly modify (6.20) as follows

$$l_{nd+2}(v^*, \nu_K) \cdot (F(v) \cdot \nu_K) = l_{nd+2}(v^*, \nu_K) \cdot (F(\underline{v}) \cdot \nu_K) \,, \tag{6.21}$$

where $v^* = (\underline{\rho}, \underline{\rho}|\underline{u}|\nu_K, \underline{\rho E})$, that is we change $\underline{v}$ in order that its velocity becomes also normal to the boundary. Let us write now explicitly (6.21) :

$$\rho\mu(\underline{c}^2 + \underline{k}|\underline{u}|^2 - |\underline{u}|\underline{c} + \underline{k}(H - \underline{H})) + (\rho\mu^2 + p)(\underline{c} - \underline{k}|\underline{u}|) = l_{nd+2}(v^*, \nu_K) \cdot (F(\underline{v}) \cdot \nu_K) \,. \tag{6.22}$$

Then we write the two supplementary boundary conditions as

$$g_1(v) = 0 \,, \quad g_2(v) = 0 \,. \tag{6.23}$$

According to Theorem 1, the local solvability of (6.22) and (6.23) is reduced to the condition :

$$det_{1 \le k,l \le 2} \left(\sum_{i=1}^{m} r_k^i(v^*, \nu_K) \frac{\partial g_l}{\partial v_i}(v^*) \right) \neq 0 \,. \tag{6.24}$$

But now since we have brought back the nd dimensional problem to a 1 dimensional one, we can use the results and the methods of Section 10.1. See in particular Propositions 4 to 6.

It remains to address the general case where the direction of the flow, although entering into Ω, is not normal to its boundary. In this case, the normal flux reads as follows :

$$F(v) \cdot \nu_K = (\rho\mu\alpha \cdot \nu_K, \rho\mu^2\alpha \cdot \nu_K \alpha + p\nu_K, \rho\mu H\alpha \cdot \nu_K) \,. \tag{6.25}$$

Here again, in order to find this normal flux, according to (3.13), we first take

$$l_{nd+2}(\underline{v}, \nu_K) \cdot (F(v) \cdot \nu_K) = l_{nd+2}(\underline{v}, \nu_K) \cdot (F(\underline{v}) \cdot \nu_K) \,, \tag{6.26}$$

as an information coming from inside Ω. Then we write the two supplementary boundary conditions as

$$g_1(v) = 0 \,, \quad g_2(v) = 0 \,. \tag{6.27}$$

According to Theorem 1, the local solvability of (6.26) and (6.27) is reduced to the condition :

$$det_{1 \le k,l \le 2} \left(\sum_{i=1}^{m} r_k^i(\underline{v}, \nu_K) \frac{\partial g_l}{\partial v_i}(\underline{v}) \right) \neq 0 \,. \tag{6.28}$$

7 A two fluid model

As already stressed in Section 4, our motivation comes from models arising in two phase fluid computational fluid dynamics. More precisely, we address the so called averaged models which can predict kinematic and thermal non-equilibrated flows. (Boure and Delhaye [6], Drew and Lahey [18], Ishii [34], Ransom [44]) These models are derived by the application of an average process (with respect to time or space or even a statistical averaging) to the classical compressible Navier-Stokes equations in each fluid or phase, separated by interfaces.

7.1 A 6 equations model

This leads to a system of 6 balance equations that reads as follows ($k = 1$ or 2) :

$$\frac{\partial(\alpha_k\rho_k)}{\partial t} + \nabla\cdot(\alpha_k\rho_k u_k) = \Gamma_k, \tag{7.1}$$

$$\frac{\partial(\alpha_k\rho_k u_k)}{\partial t} + \nabla\cdot\ \ (\alpha_k(\rho_k u_k\otimes u_k + p\mathbb{1})) - p\nabla\alpha_k = \alpha_k\rho_k g + M_k + u_{k,i}\Gamma_k, \tag{7.2}$$

$$\frac{\partial(\alpha_k\rho_k E_k)}{\partial t} + \nabla\cdot(\alpha_k\rho_k H_k u_k) + p\frac{\partial\alpha_k}{\partial t} = (\alpha_k\rho_k g + M_k)\cdot u_k + H_k\Gamma_k + Q_k, \tag{7.3}$$

This model is simplified in the sense that we have omitted in its right hand side the contributions which are related to dissipative phenomena. System (7.1)-(7.3) will be termed as the basic model.
Let us now describe the physical meaning of each variables : α_i is the volume fraction of the fluid i, ρ_i is the density of the fluid i, u_i denotes the velocity of the phase i and p is the thermodynamic pressure. Denoting by e_i the specific internal energy of the phase i, we have set $E_i = e_i + \frac{1}{2}|u|^2$: the total specific energy of the fluid i and $H_i = E_i + \frac{p}{\rho_i}$ the total specific enthalpy of the fluid i (we shall also use the notation $h_i \equiv e_i + \frac{p}{\rho_i}$ for the specific enthalpy of the fluid i), the Γ_k's denote mass transfers term with $\Gamma_1 + \Gamma_2 = 0$, the M_k's momentum transfers Q_k's heat transfers and finally the $u_{k,i}$ are interfacial velocities. Gravity is denoted by g.

We have the relation $\alpha_1 + \alpha_2 = 1$ and in order to close the system (7.1)-(7.3), we have to write two equations of state :

$$F_i(p,\rho_i,e_i) = 0\,,\quad i = 1,2\,. \tag{7.4}$$

7.2 An isentropic model

An isentropic version of this system of 6 equations can be obtained as follows. Introducing the specific entropy of the fluid i, s_i, defined by :

$$T_i ds_i = de_i - \frac{p}{\rho_i^2}d\rho_i\,, \tag{7.5}$$

and *assuming that there is no production of entropy into the shocks*, equation (7.3) leads to :

$$\frac{\partial(\alpha_k\rho_k s_k)}{\partial t} + \nabla\cdot(\alpha_k\rho_k s_k u_k) = \frac{Q_k}{T_k} + \left(\frac{u_k - u_{k,i}}{T_k}u_k + s_k\right)\Gamma_k\,, \tag{7.6}$$

In the case of absence of mass transfers between the two fluids ($\Gamma_k \equiv 0$) and of heat transfers ($Q_k \equiv 0$), these equations read as

$$\frac{\partial(\alpha_k\rho_k s_k)}{\partial t} + \nabla\cdot(\alpha_k\rho_k s_k u_k) = 0\,. \tag{7.7}$$

In view of (7.1) and (7.7), we can have solutions with constant entropies *i.e.* with s_1 and s_2 constant. The system then reduces to equations (7.1) and (7.2) and the equations of states (7.4) are replaced by isentropic ones :

$$G_i(p,\rho_i) = 0\,, i = 1,2\,. \tag{7.8}$$

For the sake of simplicity in the exposition, let us concentrate on the following simplified system :

$$\frac{\partial(\alpha_k\rho_k)}{\partial t} + \nabla\cdot(\alpha_k\rho_k u_k) = 0\,, \tag{7.9}$$

$$\frac{\partial(\alpha_k\rho_k u_k)}{\partial t} + \nabla\cdot(\alpha_k(\rho_k u_k\otimes u_k + p\mathbb{1})) - p\nabla\alpha_k = \alpha_k\rho_k g + M_k\,, \tag{7.10}$$

with the two equations of states (EOS) given by (7.8) and where we only take into account interfacial pressure in the momentum transfer. Hence the forces M_k which satisfy $M_1 + M_2 = 0$ reads

$$M_k = -(p - p_{interface})\nabla\alpha_k\,, \tag{7.11}$$

For instance, Bestion [5] has proposed to take :

$$(p - p_{interface}) \equiv \delta \frac{\alpha_1\alpha_2\rho_1\rho_2}{\alpha_1\rho_2 + \alpha_2\rho_1}(u_1 - u_2)^2 \,. \tag{7.12}$$

The order of magnitude of the parameter δ is about 1 and its role is to allow the system (7.9)-(7.10) to be hyperbolic.

This system has already the non conservative form (4.4) provided we take (here $nd = 3$ and $m = 8$) :

$$v = (\alpha_1\rho_1, \alpha_1\rho_1 u_1, \alpha_2\rho_2, \alpha_2\rho_2 u_2), \quad u_k \in \mathbb{R}^3 \,, \tag{7.13}$$

$$\begin{aligned} F(v)\cdot\nu = (&\alpha_1\rho_1(u_1\cdot\nu), \alpha_1\rho_1(u_1\cdot\nu)u_1 + \alpha_1(p-\pi)\nu, \\ &\alpha_2\rho_2(u_2\cdot\nu), \alpha_2\rho_2(u_2\cdot\nu)u_2 + \alpha_2(p-\pi)\nu) \,, \end{aligned} \tag{7.14}$$

$$\sum_{j=1}^{3} C_j(v)\frac{\partial v}{\partial x_j} = (0, -\Gamma\nabla\alpha_1, 0, -\Gamma\nabla\alpha_2) \,, \tag{7.15}$$

$$S(v) = (0, \alpha_1\rho_1 g, 0, \alpha_2\rho_2 g) \,, \tag{7.16}$$

where g denotes the gravity, $\pi = \pi(t)$ is a time-dependent function chosen in order that the Jacobian matrix $J_\nu(v)$, see (4.13), is invertible and finally we have denoted :

$$\Gamma \equiv p - \pi - \delta \frac{\alpha_1\alpha_2\rho_1\rho_2}{\alpha_1\rho_2 + \alpha_2\rho_1}(u_1 - u_2)^2 \,. \tag{7.17}$$

Remark 17 *The relation (7.15) does not give explicitly the matrices $C_j(v)$. In order to obtain these quantities, we have to compute the derivatives $\frac{\partial\alpha_k}{\partial v_j}$. This is done by using the EOS (7.8) as follows. The system allowing to compute the α_k and p from the components of v is*

$$\alpha_1 + \alpha_2 = 1 \,, \qquad G_1(p, \frac{v_1}{\alpha_1}) = 0 \,, \qquad G_2(p, \frac{v_5}{\alpha_2}) = 0 \,. \tag{7.18}$$

The differentiation of this system with respect to the v_k's will then produce a linear system for the $\frac{\partial\alpha_k}{\partial v_j}$.
For example, when one considers the mixture of an incompressible liquid (like water) and of a compressible gas (like air), a simple couple of EOS is $p = A\rho_1^\gamma$, ρ_2 =constant, where γ is a constant larger than 1. In this case the previous system can be solved explicitly :

$$\alpha_1 = \frac{\rho_2 - v_5}{\rho_2}, \qquad \alpha_2 = \frac{v_5}{\rho_2}, \qquad p = A\left(\frac{v_1\rho_2}{\rho_2 - v_5}\right)^\gamma \,. \tag{7.19}$$

In order to apply the characteristic boundary conditions (4.25), we have to construct the Jacobian matrix J_ν which reads as :

$$J_\nu(v) = A_\nu(v) - C_\nu(v) \,, \tag{7.20}$$

where $A_\nu(v) =$

$$\begin{pmatrix} 0 & \nu & 0 & 0 \\ -u_1\cdot\nu u_1 + \alpha_1 p^{(1)}\nu & u_1\otimes\nu + u_1\cdot\nu\mathbb{1} & \alpha_1 p^{(5)}\nu & 0 \\ 0 & 0 & 0 & \nu \\ \alpha_2 p^{(1)}\nu & 0 & -u_2\cdot\nu u_2 + \alpha_2 p^{(5)}\nu & u_2\otimes\nu + u_2\cdot\nu\mathbb{1} \end{pmatrix},$$

$$C_\nu(v) = -\Gamma \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha_1^{(1)}\nu & 0 & \alpha_1^{(5)}\nu & 0 \\ 0 & 0 & 0 & 0 \\ \alpha_2^{(1)}\nu & 0 & \alpha_2^{(5)}\nu & 0 \end{pmatrix}, \tag{7.21}$$

and we have denoted by $q^{(\ell)} \equiv \frac{\partial q}{\partial v_\ell}$.

7.3 The boundary condition treatment

7.3.1 Wall conditions

Let us assume that K is a control volume that meets $\partial\Omega$, ν_K is the unit external normal and $\underline{v}$ denotes v_K. On a wall, the two phase slip, it means that their normal velocity $u_1 \cdot \nu_K$ and $u_2 \cdot \nu_K$ are equal to 0. Hence the normal flux $F(v) \cdot \nu_K$ is equal to

$$F(v) \cdot \nu_K = (0, \alpha_1(p-\pi)\nu_K, 0, (1-\alpha_1)(p-\pi)\nu_K, 0).$$

Therefore, to determine this flux, on one hand we only have to find the unknowns p and α_1. On the other hand, according the wave structure obtained by Cortes [10] with a density perturbation method, two eigenvalues namely $\lambda_m(v, \nu_K)$, $\lambda_{m-1}(v, \nu_K)$ are positive. Then two information come from inside Ω and if we follow our strategy, we have to impose :

$$\begin{aligned} l_m(\underline{v}, \nu_K) \cdot (0, \alpha_1(p-\pi)\nu_K, 0, (1-\alpha_1)(p-\pi)\nu_K, 0) &= l_m(\underline{v}, \nu_K) \cdot (F(\underline{v}) \cdot \nu_K) \\ l_{m-1}(\underline{v}, \nu_K) \cdot (0, \alpha_1(p-\pi)\nu_K, 0, (1-\alpha_1)(p-\pi)\nu_K, 0) &= l_{m-1}(\underline{v}, \nu_K) \cdot (F(\underline{v}) \cdot \nu_K). \end{aligned}$$

This system can be solved and gives explicit formulas for the pressure p and the void fraction α_1.

7.3.2 A numerical illustration

Let us give now a numerical illustration based on what is known as Ransom faucet flow [44] which is a numerical benchmark. This test case is well referenced and analytically simple. It has also the great interest that it contains some of the important features in the field at least from the numerical point of view. Moreover analytical solution is available for comparison, [44]. The solution mainly consists of a shock wave that travels under the effect of gravity.

The continuous model considered in dimension 1 corresponds to the system (7.1)-(7.3) without mass transfer :

$$\frac{\partial(\alpha_1\rho_1)}{\partial t} + \frac{\partial(\alpha_1\rho_1 u_1)}{\partial x} = 0, \tag{7.22}$$

$$\frac{\partial(\alpha_2\rho_2)}{\partial t} + \frac{\partial(\alpha_2\rho_2 u_2)}{\partial x} = 0, \tag{7.23}$$

$$\frac{\partial(\alpha_1\rho_1 u_1)}{\partial t} + \frac{\partial(\alpha_1(\rho_1 u_1 \otimes u_1 + p\mathbb{1}))}{\partial x} - p\frac{\partial\alpha_1}{\partial x} = \alpha_1\rho_1 g + M_1, \tag{7.24}$$

$$\frac{\partial(\alpha_2\rho_2 u_2)}{\partial t} + \frac{\partial(\alpha_2(\rho_2 u_2 \otimes u_2 + p\mathbb{1}))}{\partial x} - p\frac{\partial\alpha_2}{\partial x} = \alpha_2\rho_2 g + M_2, \tag{7.25}$$

$$\frac{\partial(\alpha_1\rho_1 E_1)}{\partial t} + \frac{\partial(\alpha_1\rho_1 H_1 u_1)}{\partial x} + p\frac{\partial\alpha_1}{\partial t} = (\alpha_1\rho_1 g + M_1) \cdot u_1, \tag{7.26}$$

$$\frac{\partial(\alpha_2\rho_2 E_2)}{\partial t} + \frac{\partial(\alpha_2\rho_2 H_2 u_2)}{\partial x} + p\frac{\partial\alpha_2}{\partial t} = (\alpha_2\rho_2 g + M_2) \cdot u_2. \tag{7.27}$$

The EOS corresponds to the case where phase 1 is a perfect polytropic gas *i.e.* $p = (\gamma-1)\rho_1 e_1$ with $\gamma = 1.4$ and where phase 2 is incompressible *i.e.* ρ_2 is constant.

This model is of the form (4.1) and let us explain now how to recast it under the form (4.4). We start with the fact that since ρ_2 is constant, $\frac{\partial\alpha_2}{\partial t} = \frac{1}{\rho_2}\frac{\partial v_2}{\partial t}$. But now according to (7.23), $\frac{\partial v_2}{\partial t} = -\frac{\partial v_4}{\partial x}$, so that finally :

$$\frac{\partial\alpha_1}{\partial t} = \frac{1}{\rho_2}\frac{\partial v_4}{\partial x}, \frac{\partial\alpha_2}{\partial t} = -\frac{1}{\rho_2}\frac{\partial v_4}{\partial x}. \tag{7.28}$$

Hence the two terms in (7.26) and (7.27) which contributes to $D(v)\frac{\partial v}{\partial t}$ in (4.1) have been converted into terms which enters in $C(v)\frac{\partial v}{\partial x}$ in (4.4). Full details are given in Ghidaglia *et al.* [27] from which the numerical result below is taken.

All the numerical values are expressed in the International System of Units. The equations are posed for x between 0 and 12 and the gravity is taken as $g = 10$. The boundary conditions are as follows :

$$\alpha_1(0,t) = 0.2\,, u_1(0,t) = 0\,, u_2(0,t) = 10\,, \tag{7.29}$$

$$h_2(0,t) = 209\,280\,, \tag{7.30}$$

$$p(12,t) = 10^5\,, \text{ when } u_1(12,t) \geq 0\,, \tag{7.31}$$

$$p(12,t) = 10^5\,, h_1(12,t) = 324\,594\,, \text{ when } u_1(12,t) < 0\,, \tag{7.32}$$

h_2 in (7.30) corresponds to water at a temperature of 323.15 K and atmospheric pressure, while h_1 in (7.32) corresponds to air in the same conditions. Note that the h_i's denote the specific enthalpies ($h_k \equiv e_k + \frac{p}{\rho}_k$) of each fluid and should not be confused with the boundary conditions (3.10). Actually they belong to the set of physical boundary conditions (3.9).
We take $\delta = 1.01$ in the interfacial pressure term (7.11)-(7.12). This makes the system under consideration hyperbolic and as shown by Figure 3, that displays the exact solution (shock front) and the computed solutions with 768 and 1536 cells, the numerical solution is very accurate and the boundary conditions are very well captured. The higher the number of cells is, sharper is the calculated front.

In order to make a link between the physical boundary conditions (7.29) to (7.32), and our method for numerically implementing them, we discuss the eigenvalues of the system under consideration (full details are given in [27]). In one dimension, the system has 6 real eigenvalues. Two of them are u_1 and u_2 and are associated to the two global Riemann invariants s_1 and s_2 (the specific entropies). The 4 other eigenvalues are not analytically simple in terms of the physical variables. Nevertheless they can be written as $\mu - c$, μ_1, μ_2 and $\mu + c$ where $\mu + \frac{\mu_1+\mu_2}{2} = u_1 + u_2$ and indeed μ, μ_1 and μ_2 are of the same order than u_1 and u_2 while c is of the order of the speed of sound in the first phase (which here is about 300 m/s). A careful study of the behavior of these eigenvalues as various parameters vary is offered in [54] and [53] and we refer the interested reader to these papers for that matter. As it is observed numerically, in the test case considered here, 3 of these 4 eigenvalues are positive while one is negative.
So at the inlet ($x = 0$) we always have 4 eigenvalues which are positive, one which is zero and one which is negative. Hence we need according to our method 4 physical boundary conditions : (7.29)-(7.30).
At the outlet ($x = L$) there are two case during the transients.

- Either u_1 is non negative and again we have 5 eigenvalues which are non negative and one which is negative and then we need one boundary condition (observe that the normal at $x = L$ is opposite to that at $x = 0$) : (7.31).
- Or u_1 is negative and now we have 4 eigenvalues which are non negative and two which are negative and then we need two physical boundary conditions : (7.32).

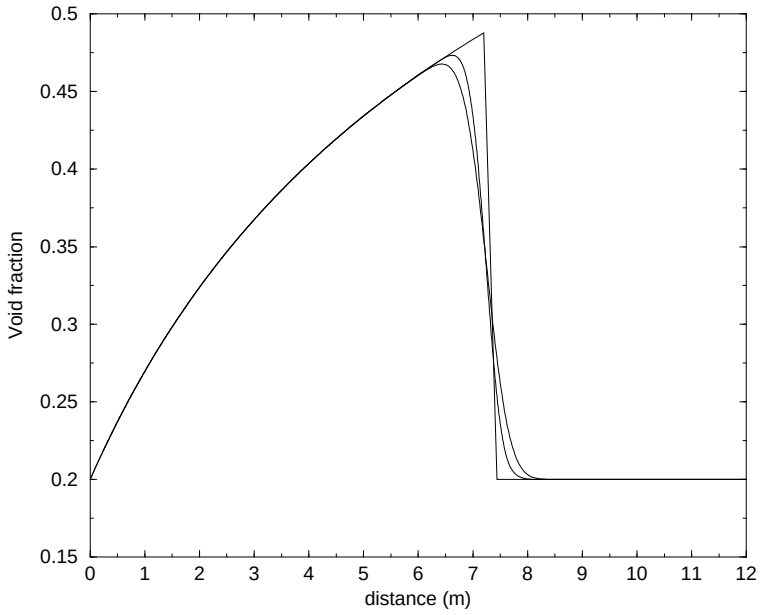

Figure 3: Exact solution, and computed ones with 768 and 1536 cells for the void fraction $\alpha_1(x,t)$ at time $t = 0.563$

8 On the numerical solution to the nonlinear equations at the boundary

We discuss in this Section a practical point of view for the implementation of the boundary condition treatment we propose in Section 3. Let us note K a control volume that meets the boundary $\partial\Omega$, ν_K the unit external normal to $K \cap \partial\Omega$, χ the number of negative eigenvalues of the Jacobian $\underline{A}_{\nu_K}$.
Let us recall that we propose to take for the numerical flux at the boundary the following form :

$$\Phi(v_K^n, K, \partial\Omega) = \Phi\,. \tag{8.1}$$

where Φ satisfies the following system :

$$\left\{ \begin{array}{l} g_k(v) = 0\,, \quad k = 1, \ldots, \chi\,, \\ l_k(\underline{v}, \nu_K) \cdot \Phi = l_k(\underline{v}, \nu_K) \cdot (F(v_K^n) \cdot \nu_K)\,, \quad k = \chi+1, \ldots, m\,, \\ \Phi = F(v) \cdot \nu_K\,. \end{array} \right. \tag{8.2}$$

The system (8.2) is solved by some few iterations of the Newton-Raphson method.

Remark 18 *In practice, system (8.2) is written in a parametric way. A more complete discussion is considered in the remaining part of this Section.*

In the following discussions, we suppose that : $\underline{v} = v_K^n$.

8.1 The case $\chi = 1$

This case corresponds in the context of example 2.1.5 to the subsonic outlet, so we present some practical tools to solve the system (3.9)-(3.10). According to Proposition 2, there is only one physical boundary condition to consider and the system (8.2) reads ($\Phi = F(v) \cdot \nu_K$) :

$$\left\{ \begin{array}{l} g_1(v) = 0 \\ l_k(\underline{v}, \nu_K) \cdot (F(v) \cdot \nu_K) = l_k(\underline{v}, \nu_K) \cdot (F(\underline{v}) \cdot \nu_K),\ k = 2, \ldots, m. \end{array} \right. \tag{8.3}$$

Since

$$\begin{aligned} F(v) \cdot \nu_K &= \sum_{k=1}^{nd+2} l_k(\underline{v}, \nu_K) \cdot (F(v) \cdot \nu_K) r_k(\underline{v}, \nu_K)\,, \\ &= F(\underline{v}) \cdot \nu_K + l_1(\underline{v}, \nu_K) \cdot (F(v) \cdot \nu_K - F(\underline{v}) \cdot \nu_K) r_1(\underline{v}, \nu_K), \end{aligned}$$

the flux $F(v) \cdot \nu_K$ can be seen as a "small" perturbation of $F(\underline{v}) \cdot \nu_K$ and system (8.3) is equivalent to the parametric system :

$$\left\{ \begin{array}{l} F(v) \cdot \nu_K = F(\underline{v}) \cdot \nu_K + \epsilon\, r_1(\underline{v}, \nu_K) \\ \epsilon \in \mathbb{R} \text{ such that } g_1(v) = 0. \end{array} \right. \tag{8.4}$$

System (8.4) can be seen as a nonlinear system of $m+1$ equations with $m+1$ unknowns (v, ϵ) which is solved by the Newton-Raphson iterative scheme.

The Newton-Raphson method. Let us present the nonlinear equations in the general form :

$$F(X) = 0 \tag{8.5}$$

The Newton-Raphson method is an iterative algorithm which start at an initial estimate X^0 of a root $X^\star$ and for which the general iterative formula is :

$$J(X^n)(X^{n+1} - X^n) = F(X^n)$$

where $J(X)$ is the Jacobian matrix of $F(X)$. Newton-Raphson algorithm's is locally quadratically convergent : only few iterations are necessary.
Now since the initial guess has to be chosen close to the exact solution, a good guess for the system (8.4) consists in taking $(v, \epsilon) \equiv (\underline{v}, 0)$. In practice, the evaluation of $J(X^n)^{-1}$ can be complicated and costly (for instance in the case of an equation of state for real gas in Example 2.1.5) and $J(X^n)$ can be replaced by some approximations : $J(X^0)$,... However convergence rate is reduced. See Stoer and Burlirsch [52] for a wider discussion.

Inviscid fluids. We now focus on the system of the Euler equations of gas dynamics. The physical boundary condition can consist in prescribing the pressure. In this particular case, the problem (8.4) reads (the pressure p is viewed as a function of state v) :

$$\left\{ \begin{array}{l} F(v)\cdot\nu_K = F(\underline{v})\cdot\nu_K + \epsilon\, r_1(\underline{v},\nu_K) \\ \epsilon\in\mathbb{R} \text{ such that } p = p_{out}. \end{array} \right. \tag{8.6}$$

Let us denote $F(\underline{v})\cdot\nu_K = (\underline{F}^1, \underline{F}, \underline{F}^{nd+2})$ and $r_1(\underline{v},\nu_K) = (\underline{R}^1, \underline{R}, \underline{R}^{nd+2})$ where $(\underline{F},\underline{R})\in\mathbb{R}^{nd}\times\mathbb{R}^{nd}$, then we get :

$$\left(\begin{array}{l} \rho(u\cdot\nu_K) \\ \rho(u\cdot\nu_K)u + p_{out}\nu_K \\ \rho H(u\cdot\nu_K) \end{array} \right) = \left(\begin{array}{c} \underline{F}^1 \\ \underline{F} \\ \underline{F}^{nd+2} \end{array} \right) + \epsilon \left(\begin{array}{c} \underline{R}^1 \\ \underline{R} \\ \underline{R}^{nd+2} \end{array} \right). \tag{8.7}$$

Using definition of the total enthalpy and introducing $(\Omega_1,\ldots,\Omega_{nd-1})$, an orthonormal basis of the hyperplane orthogonal to ν_K, (8.7) is equivalent to :

$$\left\{ \begin{array}{lcl} \rho(u\cdot\nu_K) & = & \underline{F}^1 + \epsilon\,\underline{R}^1, \\ & & \\ (\underline{F}^1 + \epsilon\,\underline{R}^1)u\cdot\nu_K + p_{out} & = & \underline{F}\cdot\nu_K + \epsilon\,\underline{R}\cdot\nu_K, \\ (\underline{F}^1 + \epsilon\,\underline{R}^1)u\cdot\Omega_1 & = & \underline{F}\cdot\Omega_1 + \epsilon\,\underline{R}\cdot\Omega_1 \ \ldots\ , \\ (\underline{F}^1 + \epsilon\,\underline{R}^1)u\cdot\Omega_{nd-1} & = & \underline{F}\cdot\Omega_{nd-1} + \epsilon\,\underline{R}\cdot\Omega_{nd-1}\,, \\ & & \\ \rho h(u\cdot\nu_K) + \frac{1}{2}u^2(\underline{F}^1 + \epsilon\,\underline{R}^1) & = & \underline{F}^{nd+2} + \epsilon\,\underline{R}^{nd+2}. \end{array} \right. \tag{8.8}$$

Since $u^2 = (u\cdot\nu_K)^2 + (u\cdot\Omega_1)^2 + \ldots + (u\cdot\Omega_{nd-1})^2$, if ρ, h, ϵ satisfy (8.6) then they necessary satisfy the system :

$$\left\{ \begin{array}{l} \rho(\underline{F}\cdot\nu_K + \epsilon\,\underline{R}\cdot\nu_K - p_{out}) = (\underline{F}^1 + \epsilon\,\underline{R}^1)^2 \\ \\ \rho h(\underline{F}\cdot\nu_K + \epsilon\,\underline{R}\cdot\nu_K - p_{out}) + \frac{1}{2}\left(\underline{F}\cdot\nu_K + \epsilon\,\underline{R}\cdot\nu_K - p_{out}\right)^2 + \\ \qquad (\underline{F}\cdot\Omega_1 + \epsilon\,\underline{R}\cdot\Omega_1)^2 + \ldots + (\underline{F}\cdot\Omega_{nd-1} + \epsilon\,\underline{R}\cdot\Omega_{nd-1})^2\big\} = \\ \qquad (\underline{F}^{nd+2} + \epsilon\,\underline{R}^{nd+2})(\underline{F}^1 + \epsilon\,\underline{R}^1)). \end{array} \right.$$

If h is a function of ρ and p, then we get a nonlinear system of 2 equations where the unknowns are ρ and ϵ. Moreover in the case of a polytropic gas, the specific enthalpy h is equal to $\frac{\gamma p}{\rho(\gamma-1)}$, and ϵ is solution of a quadratic equation :

$$\begin{array}{l} \frac{\gamma p_{out}}{\gamma-1}(\underline{F}\cdot\nu_K + \epsilon\underline{R}\cdot\nu_K - p_{out}) + \frac{1}{2}\left\{(\underline{F}\cdot\nu_K + \epsilon\,\underline{R}\cdot\nu_K - p_{out})^2 + \right. \\ \qquad \left. (\underline{F}\cdot\Omega_1 + \epsilon\,\underline{R}\cdot\Omega_1)^2 + \ldots + (\underline{F}\cdot\Omega_{nd-1} + \epsilon\,\underline{R}\cdot\Omega_{nd-1})^2\right\} = \\ \qquad (\underline{F}^{nd+2} + \epsilon\,\underline{R}^{nd+2})(\underline{F}^1 + \epsilon\,\underline{R}^1)). \end{array} \tag{8.9}$$

Once ϵ is computed with a Newton-Raphson method, relation (8.4) gives the flux at the boundary.

8.2 The case $\chi = m-1$

In this case, system (8.2) consists in finding Φ such that :

$$\left\{ \begin{array}{l} g_k(v) = 0\,, \quad k = 1,\ldots,m-1\,, \\ l_m(\underline{v},\nu_K)\cdot\Phi = l_m(\underline{v},\nu_K)\cdot(F(\underline{v})\cdot\nu_K)\,, \\ \Phi = F(v)\cdot\nu_K\,. \end{array} \right. \tag{8.10}$$

For the Euler equations of gas dynamics, it corresponds to the subsonic inlet case : see Section 6.2.2.

Inviscid fluids. Let us assume that the direction α (unit vector) of the flow is prescribed *i.e.* that velocity writes : $u = \mu\alpha_{in}$ where μ is a positive number such that $-c < \mu\alpha_{in}\cdot\nu_K < 0$. Therefore system (8.10) reduces to the form (we note $l_{nd+2}(\underline{v},\nu_K) = (\underline{L}^1, \underline{L}, \underline{L}^{nd+2})$ with $\underline{L}\in\mathbb{R}^{nd}$ and $\nu = \nu_K$) :

$$\left\{ \begin{array}{l} g_1(v) = 0\,, \\ g_2(v) = 0\,, \\ \underline{L}^1\rho\mu(\alpha_{in}\cdot\nu) + \rho\mu^2(\alpha_{in}\cdot\underline{L})(\alpha_{in}\cdot\nu) + p(\underline{L}\cdot\nu) \\ \qquad\qquad + \underline{L}^{nd+2}\rho H\mu(\alpha_{in}\cdot\nu) = l_{nd+2}(\underline{v},\nu)\cdot(F(\underline{v})\cdot\nu) \\ \Phi = F(v)\cdot\nu\,. \end{array} \right. \tag{8.11}$$

We now consider several examples of the numerical treatment at inlet boundary (see Section 12) corresponding to the two functions g_1 and g_2. These examples yield to a nonlinear scalar equation. Once this equation is solved, the state v of system (8.10) can be completely determined and furthermore the flux Φ.

Thermodynamics variables :

When 2 thermodynamics variables are prescribed, from remark 20 all the other ones are determined. Let us therefore denote ρ_{in}, p_{in}, h_{in} the prescribed density, pressure and specific enthalpy. Then the problem (8.11) reduces to solve in μ the following nonlinear equation :

$$\begin{aligned}\underline{L}^{nd+2}\rho_{in}(\alpha_{in}\cdot\nu)\mu^3+\rho_{in}(\alpha_{in}\cdot\underline{L})(\alpha_{in}\cdot\nu)\mu^2\\ +(\underline{L}^1\rho_{in}(\alpha_{in}\cdot\nu)+\underline{L}^{nd+2}\rho_{in}h_{in}(\alpha_{in}\cdot\nu))\mu\\ +p_{in}(\underline{L}\cdot\nu)=l_{nd+2}(\underline{v},\nu)\cdot(F(\underline{v})\cdot\nu)\,.\end{aligned}\tag{8.12}$$

Internal energy and velocity :

We assume that $g_1(v)=e-e_{in}$ and $g_2(v)=\mu-\mu_{in}$. From e_{in} and the equation of state, the pressure p and the specific enthalpy h can be viewed as functions of ρ. This implies that ρ is solution of :

$$\begin{aligned}\underline{L}^1\rho\mu_{in}(\alpha_{in}\cdot\nu)+\rho\mu_{in}^2(\alpha_{in}\cdot\underline{L})(\alpha_{in}\cdot\nu)+p(\rho)(\underline{L}\cdot\nu)\\ +\underline{L}^{nd+2}\rho(h(\rho)+\frac{1}{2}\mu_{in}^2)\mu_{in}(\alpha_{in}\cdot\nu)=l_{nd+2}(\underline{v},\nu)\cdot(F(\underline{v})\cdot\nu)\,.\end{aligned}\tag{8.13}$$

Internal energy and mass flow :

We assume that $g_1(v)=e-e_{in}$ and $g_2(v)=\rho\mu-(\rho\mu)_{in}$. As in the previous example, p and h can be viewed as functions of ρ. This implies that ρ is solution of :

$$\begin{aligned}\underline{L}^1(\rho\mu)_{in}(\alpha_{in}\cdot\nu)+\frac{(\rho\mu)_{in}^2}{\rho}(\alpha_{in}\cdot\underline{L})(\alpha_{in}\cdot\nu)+p(\rho)(\underline{L}\cdot\nu)\\ +\underline{L}^{nd+2}(h(\rho)+\frac{(\rho\mu)_{in}^2}{2\rho^2})(\rho\mu)_{in}(\alpha_{in}\cdot\nu)=l_{nd+2}(\underline{v},\nu)\cdot(F(\underline{v})\cdot\nu)\,.\end{aligned}\tag{8.14}$$

Entropy and total enthalpy :

From the equation of state and from $s=s_{in}$ and $h=H_{in}-\frac{1}{2}\mu^2$, density ρ and pressure p can be viewed as functions of μ. Then we obtain a nonlinear equation in μ :

$$\begin{aligned}\underline{L}^1\rho(\mu)\mu(\alpha_{in}\cdot\nu)+\rho(\mu)\mu^2(\alpha_{in}\cdot\underline{L})(\alpha_{in}\cdot\nu)+p(\mu)(\underline{L}\cdot\nu)\\ +\underline{L}^{nd+2}\rho(\mu)H_{in}\mu(\alpha_{in}\cdot\nu)=l_{nd+2}(\underline{v},\nu)\cdot(F(\underline{v})\cdot\nu)\,.\end{aligned}\tag{8.15}$$

9 Applications : A numerical comparison of boundary treatments

9.1 Perfect gas dynamics : the two-dimensional sinus bump benchmark

To illustrate the efficiency of the discretization of the boundary conditions we discussed in Section 3, we consider different treatments of the inlet and outlet boundaries for the two-dimensional flow of a perfect gas in a channel with a thick sinus bump on the lower wall. This test was originally proposed at the 1979 GAMM Workshop [45]. Let us define the geometry : the computational domain is the rectangle $[0,3]\times[0,1]$ with a 10% high sinus curve located in the arc $[0,1]$. Since the non stationary problem is concerned, an initial condition has to be defined. A uniform one is computed on the computational mesh from the non dimensional free stream variables and from the imposed "inlet" Mach number : $\mathcal{M}_0$. The free stream is defined from the speed of sound ($c_0=1$), the module of velocity ($|u_0|=\mathcal{M}_0$), the pressure ($p_0=1$), and the direction of the flow (the unit vector $\alpha_0=(1,0)$). Internal energy, density and temperature are obtained from the equation of state for a polytropic gas and from the Joule-Thompson relation. We get ($R=2/5$ and $\gamma=7/5$) :

$$\left\{\begin{array}{rcl}T_0&=&\dfrac{c_0^2}{\gamma R}\\ e_0&=&\dfrac{R}{\gamma-1}T_0\\ \rho_0&=&\dfrac{p_0}{(\gamma-1)e_0}\end{array}\right.\quad\Rightarrow\quad\left\{\begin{array}{rcl}T_0&=&\dfrac{1}{\gamma R}\\ e_0&=&\dfrac{1}{\gamma(\gamma-1)}\\ \rho_0&=&\gamma\end{array}\right.$$

When the inlet Mach number is less than 0.6, the flow is subsonic in all the domain and the solution converge to a stationary isentropic flow. Then three types of boundary conditions have to be considered : subsonic inlet, subsonic outlet and wall conditions.

In the present numerical experiments, we consider on the upper and lower wall the numerical flux described in Section 6.1 :

$$\Phi(\underline{v}, K, \partial\Omega) = (0, \frac{l_{nd+2}(\underline{v}, \nu) \cdot (F(\underline{v}) \cdot \nu)}{l_{nd+2}(\underline{v}, \nu) \cdot (0, \nu, 0)} \nu, 0) .$$

In the two first Section, we describe two numerical treatments of the boundary conditions that are implemented by first constructing a boundary face state $V_b = (\rho_b, \rho_b u_b, \rho_b E_b)$ and then by computing the VFFC flux at the boundary from this state and the interior one, following definition 2 and 3.

9.1.1 The Riemann invariant boundary conditions : Rinv

We consider here boundary conditions treatment based on the Riemann invariants defined in Section 3.3 and computed for the Euler Equations in Section 10. We refer to Hirsch [31] and reference therein, and Darmofal [12]. For the subsonic inlet boundary, they read (since ν is the outer unit normal, there is a minus sign) :

$$\begin{cases} -u_b \cdot \nu + \dfrac{2}{\gamma - 1} c_b = -u_0 \cdot \nu + \dfrac{2}{\gamma - 1} c_0 \\ \dfrac{p_b}{\rho_b^\gamma} = \dfrac{p_0}{\rho_0^\gamma} \\ -u_b \cdot \nu - \dfrac{2}{\gamma - 1} c_b = -\underline{u} \cdot \nu - \dfrac{2}{\gamma - 1} \underline{c} \\ \alpha_b = \alpha_0 \end{cases} \tag{9.1}$$

where the subscript 0 correspond to the free stream variables. The first relation corresponds to the Riemann invariant along the incoming characteristic associated to the eigenvalue $\underline{u} \cdot \nu - \underline{c}$. The second relation means that the entropy is constant along the characteristic associated to $\underline{u} \cdot \nu$. These two relations deal with physical boundary conditions and they are estimated from the free stream state. The third relation which is associated with a numerical boundary condition, is estimated from inside the domain and corresponds to the Riemann invariant along the outcoming characteristic associated to the eigenvalue $\underline{u} \cdot \nu + \underline{c}$. The last relation on the direction of the flow is the additional condition for the two-dimensional boundary implementation.
For the subsonic outlet boundary, the boundary state is obtained from :

$$\begin{cases} u_b \cdot \nu + \dfrac{2}{\gamma - 1} c_b = u_0 \cdot \nu + \dfrac{2}{\gamma - 1} c_0 \\ \dfrac{p_b}{\rho_b^\gamma} = \dfrac{\underline{p}}{\underline{\rho}^\gamma} \\ u_b \cdot \nu + \dfrac{2}{\gamma - 1} c_b = \underline{u} \cdot \nu + \dfrac{2}{\gamma - 1} \underline{c} \\ u_b \cdot \tau = \underline{u} \cdot \tau \end{cases} \tag{9.2}$$

where τ is a normalized vector orthogonal to ν. The first relation comes from the incoming, negative characteristic associated to the eigenvalue $\underline{u} \cdot \nu - \underline{c}$. The three others conditions are estimated from the interior since they are associated to the positive eigenvalue, *i.e.* the outcoming characteristics.

9.1.2 The partial Riemann problem treatment of boundary conditions : PR_QT

We consider in this section, the incomplete Riemann problem technique proposed by Dubois and LeFloch [20] and described in Section 3.3.2.
For subsonic inflow, the state V_b is the unique state which belongs to the boundary "manifold" *i.e.* that respects given boundary conditions and which can be connected to the interior state through two simple waves. Let's suppose that a direction of the flow α_0, a mass flow Q_0 and a temperature T_0 are given then the resolution of the boundary state can be conducted in the plane of normal velocity and pressure since from $\rho_b u_b \cdot \nu = Q_0$ and from definition of internal energy $e_b = \frac{R}{\gamma - 1} T_0$, the pressure $p_b = (\gamma - 1)\rho_b e_b$ is expressed in terms of $u_b \cdot \nu$ with the hyperbola that defines the boundary "manifold" on the normal velocity-pressure plane :

$$p_b = \frac{R T_0 Q_0}{u_b \cdot \nu} \tag{9.3}$$

The additional condition for the two-dimensional case is simply : $\alpha_b = \alpha_0$. Therefore, the boundary state V_b which satisfies this hyperbola is connected to an intermediate state μ_1 through a 2-simple wave and this intermediate state is connected to the interior state $\underline{v}$ through a 1-simple wave. If the 1-wave is a 1-rarefaction

wave, then the partial Riemann problem reduces to the solution of the following system :

$$\left\{\begin{array}{l} \underline{u}\cdot\nu+\dfrac{2}{\gamma-1}\underline{c}=u_1\cdot\nu+\dfrac{2}{\gamma-1}c_1 \\ \dfrac{\underline{p}}{\underline{\rho}^\gamma}=\dfrac{p_1}{\rho_1^\gamma} \\ u_1\cdot\nu=u_b\cdot\nu \\ p_1=p_b \end{array}\right. \tag{9.4}$$

From this system, the normal component of velocity can be finally made explicit as solution of :

$$u_b\cdot\nu=-\frac{2\sqrt{\gamma}}{\gamma-1}\left(\frac{RT_0Q_0}{u_b\cdot\nu}\right)^{\frac{\gamma-1}{2\gamma}}\left(\frac{\underline{p}}{\underline{\rho}^\gamma}\right)^{\frac{1}{2\gamma}}+\underline{u}\cdot\nu+\frac{2}{\gamma-1}\underline{c} \tag{9.5}$$

In the case of a 1-shock wave, the partial Riemann problem reduces to :

$$\left\{\begin{array}{l} u_1\cdot\nu-\underline{u}\cdot\nu+\sqrt{\dfrac{2}{\underline{\rho}((\gamma+1)p_1+(\gamma-1)\underline{p})}}\,(p_1-\underline{p})=0 \\ u_1\cdot\nu=u_b\cdot\nu \\ p_1=p_b \end{array}\right. \tag{9.6}$$

and in consequence that $u_b\cdot\nu$ is solution of :

$$u_b\cdot\nu-\underline{u}\cdot\nu+\sqrt{\frac{2}{\underline{\rho}((\gamma+1)\dfrac{RT_0Q_0}{u_b\cdot\nu}+(\gamma-1)\underline{p})}}\,(\frac{RT_0Q_0}{u_b\cdot\nu}-\underline{p})=0 \tag{9.7}$$

Problems (9.5) and (9.7) are in practice solved with some few iterations of Newton.
For the outflow subsonic condition, the unique state associated to the prescribed pressure p_0 and that can be connected to the interior state by a 1-wave is exactly solved thanks to the relations :

$$\left\{\begin{array}{l} p_b=p_0 \\ \dfrac{p_b}{\rho_b^\gamma}=\dfrac{\underline{p}}{\underline{\rho}^\gamma} \\ u_b\cdot\nu+\dfrac{2}{\gamma-1}c_b=\underline{u}\cdot\nu+\dfrac{2}{\gamma-1}\underline{c} \\ u_b\cdot\tau=\underline{u}\cdot\tau \end{array}\right. \tag{9.8}$$

in the case of a 1-rarefaction wave and thanks to :

$$\left\{\begin{array}{l} p_b=p_0 \\ \dfrac{p_b}{\rho_b^\gamma}=\dfrac{\underline{p}}{\underline{\rho}^\gamma} \\ u_b\cdot\nu=\underline{u}\cdot\nu-\sqrt{\dfrac{2}{\underline{\rho}((\gamma+1)p_b+(\gamma-1)\underline{p})}}\,(p_b-\underline{p}) \\ u_b\cdot\tau=\underline{u}\cdot\tau \end{array}\right. \tag{9.9}$$

in the case of a 1-shock wave.

9.1.3 The mirror treatment : Mi_QT

The last approach to impose boundary conditions differs from the two previous one. It consists in using a "mirror" ("virtual" or "ghost") cell which is placed outside the domain and where a virtual state is evaluated from given data and extrapolated values from the interior. The interface is then numerically treated as an interior one (the numerical flux is defined is Section 2.3.2).
For the subsonic inflow, the virtual state is computed from the mass flow Q_0, the temperature T_0 and the direction of the flow α_0 and from the density extrapolated from the interior. For the subsonic outflow, a state on the mirror cell is evaluated from the given pressure and from density and velocity components extrapolated from the interior.

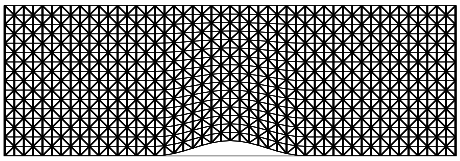

Figure 4: S1 : Structured triangulation of the channel

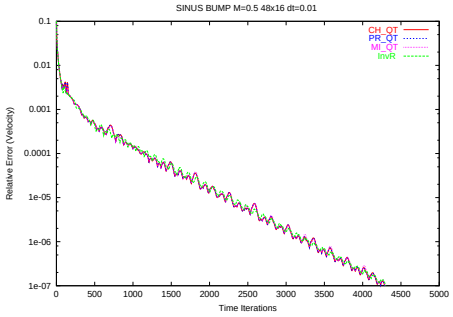

Figure 5: S1 : The L^∞ norm of the velocity residual

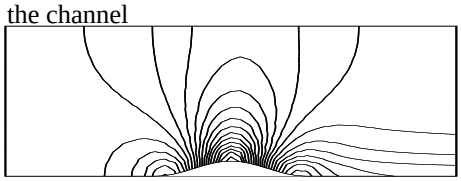

Figure 6: S1 : mach isolines for CH_QT

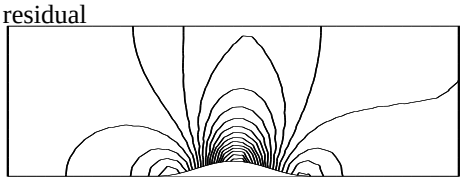

Figure 7: S1 : pressure isolines for CH_QT

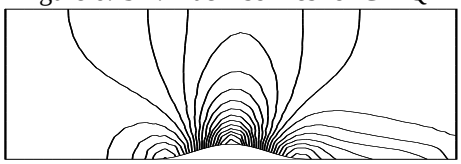

Figure 8: S1 : mach isolines for Rinv

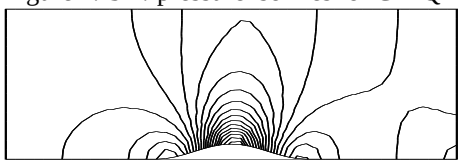

Figure 9: S1 : pressure isolines for Rinv

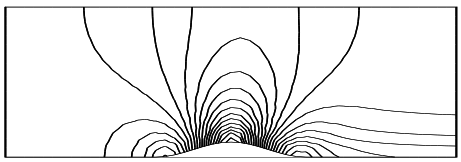

Figure 10: S1 : mach isolines for Mi_QT

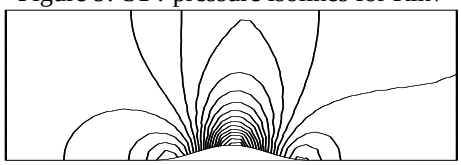

Figure 11: S1 : pressure isolines for Mi_QT

9.1.4 Numerical results for $\mathcal{M}_0 = 0.5$

The tests presented here concern a structured triangulation of the domain (see Figure 4) and an unstructured triangulation (see Figure 12) obtained by the mesh software "EMC2" developed by Hecht and Saltel [30]. All the computations are carried out with the Euler's explicit time scheme with a Courant number of 0.9. We compare the convergence to the steady solution obtained with

- the Riemann invariant boundary condition (Rinv),
- the partial Riemann invariant problem treatment of the boundary condition (PR_QT),
- the mirror treatment (MI_QT)
- the present boundary condition (CH_QT) where at the subsonic inlet boundary the flow direction $\alpha = 0$, the flow mass $Q = Q_0$ and the temperature $T = T_0$ are prescribed. Since for a polytropic gas, it is equivalent to prescribe the internal energy, we know from Proposition 6 that for these physical conditions, hypothesis of Theorem 1 are available and we can apply formula (8.14). At the outlet boundary, pressure $p = p_0$ is prescribed following formula (8.9).

As can be seen in Figure 5 for the structured mesh and in Figure 13 for the unstructured mesh, the evolution of L^∞ norms of the velocity residual (ie relative error) are quite identical, that is the four methods converge to the steady state in a similar way. Let us remark that numerical results for (PR_QT) and (CH_QT) are rigorously identical (to within rounding errors) and confirm the conclusion of the Section 3.3. In addition, when one compares steady solutions through the isoMach distributions (Figures 6, 8, 10, 14, 16, 18) or the isobars (Figures 7,

9, 11, 15, 17, 19), one can see that the computed solutions are identical in spite of slight deformations. Finally, one can observe that the present method is insensitive to the mesh structure.

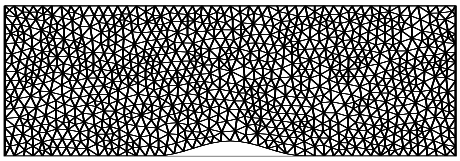

Figure 12: S2 : Unstructured triangulation of the channel

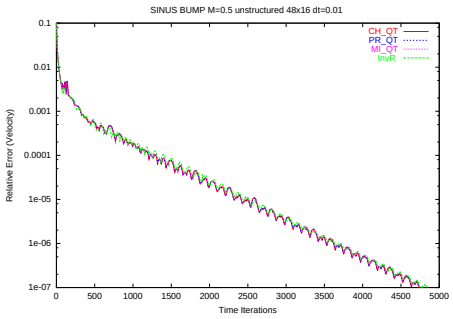

Figure 13: S2 : The L^∞ norm of the velocity residual

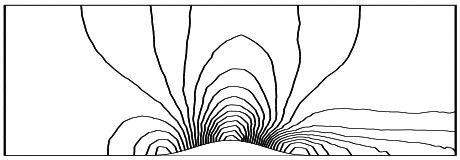

Figure 14: S2 : mach isolines for CH_QT

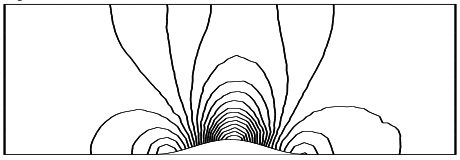

Figure 15: S2 : pressure isolines for CH_QT

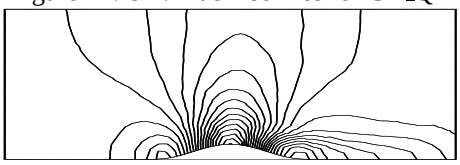

Figure 16: S2 : mach isolines for Rinv

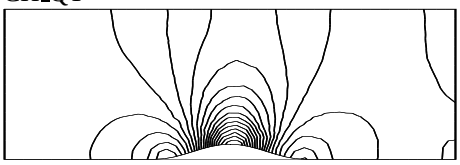

Figure 17: S2 : pressure isolines for Rinv

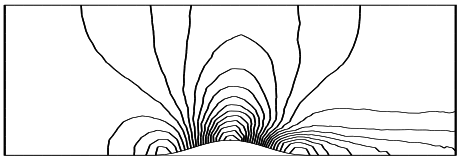

Figure 18: S2 : mach isolines for Mi_QT

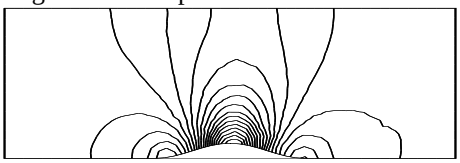

Figure 19: S2 : pressure isolines for Mi_QT

9.1.5 Behavior at low Mach numbers

It is well known that it is a difficult task to compute the solution of compressible equations for low Mach number. The main difficulty comes from the large disparity of the wave speeds (acoustic waves and waves convected at the fluid speed). Even if the incompressible equations approximate the compressible equations when the Mach number becomes small, there are numerous reasons for developing some compressible codes to solve nearly incompressible flow (see for instance Turkel *et al* [58]). Local preconditioning has been successfully used to improve convergence rates and accuracy of numerical approximations for low speed flow : Tukel [56], Turkel [57], Van Leer *et al* [59], Choi and Merkle [9], Lee [37] and references therein. The local preconditioning modifies time-dependent equations by multiplying local vector of time derivatives by a locally-evaluated, non-singular matrix. More precisely, the semi-discretization (2.64) is replaced by the following preconditioned formulation :

$$vol(K)P^{-1}\frac{dv_K}{dt} + \sum_{L\in\mathcal{N}(K)} area(K\cap L)\Phi(v_K, v_L; K, L) = 0\,. \tag{9.10}$$

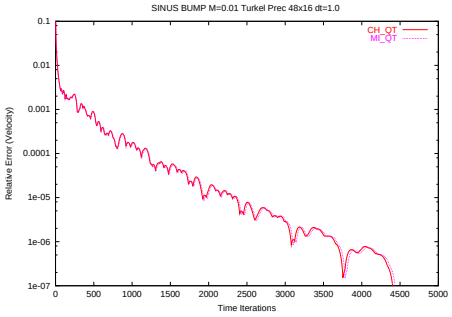

Figure 20: S1 : The L^∞ norm of the velocity residual

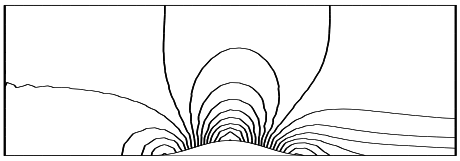

Figure 21: S1 : mach isolines for CH_QT

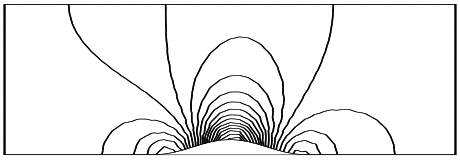

Figure 22: S1 : pressure isolines for CH_QT

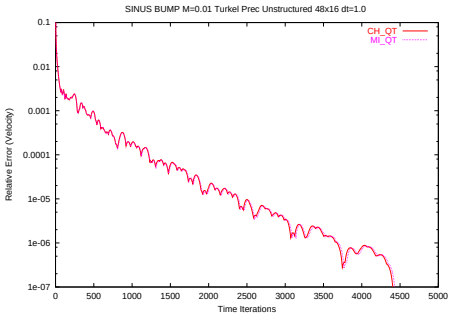

Figure 23: S2 : The L^∞ norm of the velocity residual

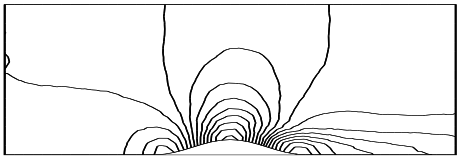

Figure 24: S2 : mach isolines for CH_QT

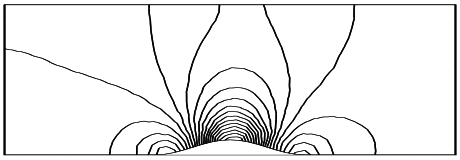

Figure 25: S2 : pressure isolines for CH_QT

This matrix P depends on the local value v_K and even if the scheme is no longer time consistent, the choice of the Turkel's diagonal preconditioner that reads (in (p, u, s) variables) :

$$P^{-1} = \begin{pmatrix} \frac{1}{\beta^2} & 0 \\ 0 & \mathbf{1}_{nd+1} \end{pmatrix} \quad \text{with} \quad \beta = \begin{cases} \epsilon & \text{if } (M) < \epsilon \\ (M) & \text{if } \epsilon < (M) < 1 \\ 1 & \text{if } (M) \geq 1 \end{cases} \tag{9.11}$$

improves the convergence and the accuracy of the steady-state discretization. Let us note that, following the observations of Van Leer *et al* [59], the numerical fluxes defined in Section 2.3.2 is also modified by the preconditioner The purpose of our computations here is not to analyze the effect of preconditioning on the boundary conditions we have previously described (as it is done in Darmofal *et al*[12]) but our aim is to observe how behave from a practical point of view the different treatments of boundary conditions when the upwind method and the time-derivative are modified by the low Mach preconditioner. The treatment by the Riemann

invariants (Rinv) and by the partial Riemann problem (PR_QT) yield unstable schemes unable to converge to the steady state unlike the mirror treatment (MI_QT) and the present method (CH_QT) as it can be observed in Figure 20 and in Figure 23 where the evolution of velocity residual for the structured mesh and for the unstructured mesh are shown. It was an expected result for the mirror treatment since the flux at the boundary is computed like an interior interface flux once the "ghost cell" volume has been evaluated. In the same way, the result for (Rinv) and (PR_QT) can be at least formally explained since the invariants and the resolution of the partial Riemann problem have not been adapted to the preconditioned equations. Figures 21 and 24 show the Mach distribution and Figures 22 and 25 the pressure distribution for $(M)_0 = 0.01$ and the accuracy of numerical solutions is verified by comparisons with solutions for $(M)_0 = 0.5$. The automatic adaptation to the Turkel's preconditioning of the present method for computing the boundary flux confirms its robustness.

9.2 Perfect gas dynamics : the "Anderson" subsonic nozzle

We consider in this example, the convergence to the steady, isentropic flow through a convergent-divergent nozzle and with a perfect gas as described in Anderson [1]. The nozzle is specified with the following area distribution :

$$A = \begin{cases} 1 + 2.2(x - 1.5)^2 & \text{for} \quad 0 \leq x \leq 1.5 \\ 1 + 0.2223(x - 1.5)^2 & \text{for} \quad 1.5 \leq x \leq 3.0 \end{cases}$$

The flow at the inlet comes from a reservoir where the internal energy ($e_0 = 1$), the density ($\rho_0 = 1$) and the direction of the flow ($\alpha_0 = (1, 0)$) are given and where the velocity is considered equal to zero. Total or stagnation temperature (T_0), total or stagnation pressure (p_0) and from an equivalent point of view enthalpy (H_0) and entropy (S_0) are evaluated from the equation of state for a perfect gas with $R = 2/5$ and $\gamma = 7/5$ and they can be expressed as :

$$\begin{cases} T_0 = \dfrac{p_0}{R\rho_0} \\ p_0 = (\gamma - 1)\rho_0 e_0 \\ S_0 = \dfrac{p_0}{\rho_0^\gamma} \\ H_0 = e_0 + \dfrac{p_0}{\rho_0} \end{cases} \Rightarrow \begin{cases} T_0 = \dfrac{\gamma - 1}{R} \\ p_0 = \gamma - 1 \\ S_0 = 1 \\ H_0 = \gamma \end{cases}$$

The exit pressure p_e is only slightly smaller than the reservoir pressure :

$$\kappa \equiv \frac{p_e}{p_0} = 0.93.$$

In a such case, the Mach number increases with distance until the minimum area cross section (the maximum reached by Mach number is less than 1) and decreases in the divergent section.

9.2.1 The partial Riemann problem treatment of boundary conditions : PR_HS

Since the enthalpy, the entropy and the direction of the flow are given at the subsonic inlet, the unique state V_b that can be connected to the interior through two simple-waves and which lies in the boundary manifold (see Dubois [19]) satisfies the relations :

$$\begin{cases} \alpha_b = \alpha_0 \\ \dfrac{u_b^2}{2} + \dfrac{\gamma p_b}{(\gamma - 1)\rho_b} = H_0 \\ \dfrac{p_b}{\rho_b^\gamma} = S_0 \end{cases} \tag{9.12}$$

This means that in the velocity module - pressure plane, we set :

$$\frac{u_b^2}{2} + \frac{\gamma}{\gamma - 1} p_b^{\frac{\gamma-1}{\gamma}} S_0^{\frac{1}{\gamma}} = H_0 \tag{9.13}$$

The boundary state is then obtained by solving a partial Riemann problem which reduces to find the state satisfying equation (9.13) and issued from an intermediate state μ_1 through a 2-wave, this state being issued

from the interior state through a 1-wave. If the 1-wave is a 1-rarefaction wave, we get :

$$\left\{\begin{array}{l} \underline{u}\cdot\nu+\dfrac{2}{\gamma-1}\underline{c}\equiv\underline{R^+}=u_1\cdot\nu+\dfrac{2}{\gamma-1}c_1 \\ \dfrac{\underline{p}}{\underline{\rho}^\gamma}\equiv\underline{S}=\dfrac{p_1}{\rho_1^\gamma} \\ u_1\cdot\nu=u_b\cdot\nu \\ p_1=p_b \end{array}\right. \tag{9.14}$$

From this system, the module of velocity can be made explicit as solution of a quadratic equation :

$$\begin{array}{l} \left(\frac{1}{2}+\frac{\gamma-1}{4}\left(\frac{S_0}{\underline{S}}\right)^{\frac{1}{\gamma}}(\alpha_b\cdot\nu)^2\right)u_b^2-\underline{R^+}\frac{\gamma-1}{2}\left(\frac{S_0}{\underline{S}}\right)^{\frac{1}{\gamma}}(\alpha_b\cdot\nu)\,u_b+ \\ \frac{\gamma-1}{4}\left(\frac{S_0}{\underline{S}}\right)^{\frac{1}{\gamma}}\underline{R^+}^2=H_0 \end{array} \tag{9.15}$$

In case of a 1-shock wave, the partial Riemann problem yields :

$$\left\{\begin{array}{l} u_1\cdot\nu-\underline{u}\cdot\nu+\sqrt{\dfrac{2}{\underline{\rho}((\gamma+1)p_1+(\gamma-1)\underline{p})}}\,(p_1-\underline{p})=0 \\ u_1\cdot\nu=u_b\cdot\nu \\ p_1=p_b \end{array}\right. \tag{9.16}$$

Then we obtain that the pressure follows :

$$\begin{array}{l} 2(H_0-\dfrac{\gamma}{\gamma-1}p_b^{\frac{\gamma-1}{\gamma}}S_0^{\frac{1}{\gamma}})= \\ \qquad\left(\underline{u}\cdot\nu-\sqrt{\frac{2}{\underline{\rho}((\gamma+1)p_b+(\gamma-1)\underline{p})}}\,(p_b-\underline{p})\right)^2 \end{array} \tag{9.17}$$

For the outlet subsonic condition, the unique state associated to the prescribed pressure p_e and that can be connected to the interior state by a 1-wave is exactly solved thanks to the relations (1-rarefaction wave) :

$$\left\{\begin{array}{l} p_b=\kappa p_0 \\ \dfrac{p_b}{\rho_b^\gamma}=\dfrac{\underline{p}}{\underline{\rho}^\gamma} \\ u_b\cdot\nu+\dfrac{2}{\gamma-1}c_b=\underline{u}\cdot\nu+\dfrac{2}{\gamma-1}\underline{c} \\ u_b\cdot\tau=\underline{u}\cdot\tau \end{array}\right.$$

or (1-shock wave) :

$$\left\{\begin{array}{l} p_b=\kappa p_0 \\ \dfrac{p_b}{\rho_b^\gamma}=\dfrac{\underline{p}}{\underline{\rho}^\gamma} \\ u_b\cdot\nu=\underline{u}\cdot\nu-\sqrt{\dfrac{2}{\underline{\rho}((\gamma+1)p_b+(\gamma-1)\underline{p})}}\,(p_b-\underline{p}) \\ u_b\cdot\tau=\underline{u}\cdot\tau \end{array}\right.$$

9.2.2 The mirror treatment : Mi_HS

This set of boundary conditions consists in the specification of the total pressure, total temperature and the direction of the flow for the inlet case. A virtual state is computed in a "mirror" cell from these values and from Mach number extrapolated from the interior. The outlet treatment is similar to the sinus bump benchmark

9.2.3 Numerical results

Figure 26 show the unstructured triangulation used for the present tests. All the computations are carried out with the Euler's explicit time scheme with a Courant number of 0.9. We compare the convergence to the steady solution obtained with

- the partial Riemann invariant problem treatment of the boundary condition (PR_HS),
- the mirror treatment (MI_HS)

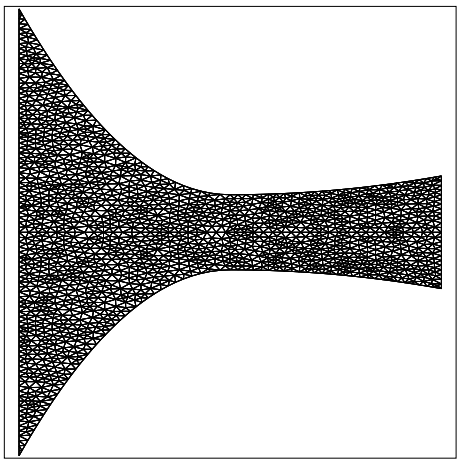

Figure 26: Unstructured triangulation of the nozzle

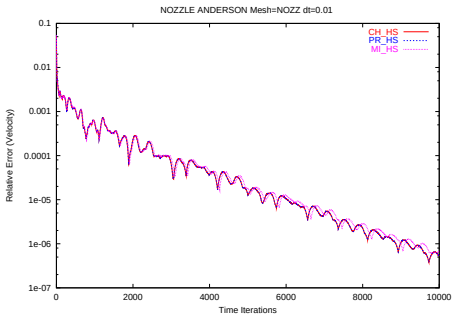

Figure 27: The L^∞ norm of the velocity residual

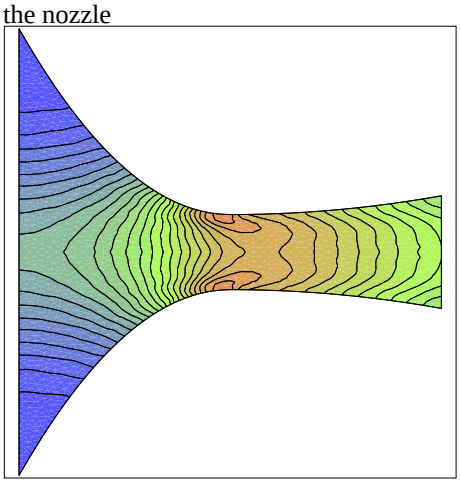

Figure 28: Mach isolines for CH_PT

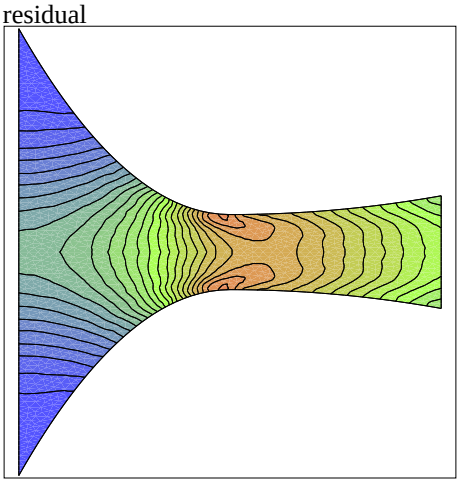

Figure 29: Mach isolines for Mi_PT

- the present boundary condition (CH_HS) where at the subsonic inlet boundary the flow direction $\alpha = 0$, the total enthalpy $H = H_0$ and the entropy $S = S_0$ are prescribed. In Proposition 7, it is proved that for these physical boundary conditions, hypothesis of Theorem 1 are available and we can apply formula (8.14). At the outlet boundary, pressure $p = \kappa p_0$ is prescribed following formula (8.9).

Figure 27 show the evolution of L^∞ norms of the velocity residual (ie relative error) for the three methods : one can see that they converge to the steady state in a similar way. Let us remark that numerical results for (PR_HS) and (CH_HT) are rigorously identical (to within rounding errors) and confirm the conclusion of the Section 3.3. In addition, when one compares steady solutions through the isoMach distributions (Figures 28, 29) one can see that the computed solutions are identical in spite of slight deformations near the smallest cross section.

10 The one dimensional case for the Euler equations

In the one dimensional case, equation (2.1) reads

$$\frac{\partial v}{\partial t} + \frac{\partial f(v)}{\partial x} = 0\,, \tag{10.1}$$

where f maps $G \subset \mathbb{R}^m$ into $\mathbb{R}^m$. We denote here by $A(v)$ the jacobian matrix $\frac{\partial f(v)}{\partial v}$. We assume that (10.1) is hyperbolic (see Definition 1) and introduce an eigensystem of $A(v)$ composed of

- the set of the real eigenvalues : $\lambda_1(v) \leq \ldots \leq \lambda_m(v)$,

- a set $(l_1(v), \dots, l_m(v))$ of left eigenvectors satisfying :

$$ {}^t A(v)\, l_k(v) = \lambda_k(v) l_k(v), \quad \text{for } k = 1, \dots, m, $$

- a set $(r_1(v), \dots, r_m(v))$ of right eigenvectors satisfying :

$$ A(v)\, r_k(v) = \lambda_k(v) r_k(v), \quad \text{for } k = 1, \dots, m. $$

and the following normalization : $(k, p = 1, \dots, m)$

$$ l_k(v) \cdot r_p(v) = \delta_{k,p}\,. $$

From the computational point of view, the major change between the multidimensional case and the one dimensional one comes from the fact that the geometrical structure in the latter is trivial.

We are only going to discuss the *Euler equations for inviscid fluids*. We have $nd = 1$ ($u \in \mathbb{R}$) and $m = 3$. The unknown is :

$$ v = (\rho, \rho u, \rho E) \tag{10.2} $$

and the flux is defined by :

$$ f(v) = (\rho u, \rho u^2 + p, \rho u H)\,. \tag{10.3} $$

Writing this equation in a quasilinear form, we have

$$ \frac{\partial v}{\partial t} + A(v)\frac{\partial v}{\partial x} = 0\,, \tag{10.4} $$

where the matrix A is simply :

$$ A(v) = \begin{pmatrix} 0 & 1 & 0 \\ K - u^2 & (2-k)u & k \\ u(K-H) & H - ku^2 & (1+k)u \end{pmatrix}. \tag{10.5} $$

The eigenvalues of $A(v)$ are

$$ \lambda_1(v) = u - c, \quad \lambda_2 = u, \quad \lambda_3 = u + c, \tag{10.6} $$

and we can take as associated eigenvectors :

$$ r_1(v) = (1, u - c, H - uc), \tag{10.7} $$
$$ r_2(v) = (1, u, H - c^2/k), \tag{10.8} $$
$$ r_3(v) = (1, u + c, H + uc). \tag{10.9} $$

The dual basis of the $(r_k(v))$ is then :

$$ l_1(v) = \frac{1}{2c^2}(K + uc, -ku - c, k), \tag{10.10} $$
$$ l_2(v) = \frac{k}{c^2}(H - u^2, u, -1), \tag{10.11} $$
$$ l_3(v) = \frac{1}{2c^2}(K - uc, -ku + c, k). \tag{10.12} $$

10.1 The case of a subsonic inlet boundary condition

We assume that the problem is posed on the interval $]0, L[$ and that the inlet boundary is held at $x = L$ in order to x in equation (10.4) coincides with the coordinate along the outer normal. We have $-c < u < 0$ and therefore with the notations of Section 3.1.3, $\chi = 2$ since there are two negative eigenvalues : $u - c$ and u. Hence we have to prescribe two boundary conditions that we write

$$ g_1(v) = 0\,, \quad g_2(v) = 0\,. \tag{10.13} $$

In this case we can write the two differential forms dg_1 and dg_2 as

$$ dg_q = \sum_{k=1}^{m} (r_k(\underline{v}) \cdot dg_k) l_k(\underline{v})\,, \quad q = 1, 2\,, \tag{10.14} $$

so that the condition (3.15) reads in the dual basis $(l_1(\underline{v}), l_2(\underline{v}), l_3(\underline{v}))$:

$$det(dg_1(\underline{v}), dg_2(\underline{v}), l_3(\underline{v})) \neq 0\,. \tag{10.15}$$

Finally, we can formulate the following proposition :

Proposition 3 *The physical boundary conditions $g_1(v) = 0$ and $g_2(v) = 0$ can be prescribed at a subsonic inlet if the condition*

$$l_3(\underline{v}) \cdot (dg_1(\underline{v}) \wedge dg_2(\underline{v})) \neq 0\,, \tag{10.16}$$

is satisfied.

Remark 19 *Since $l_3(\underline{v}) = \frac{1}{2\underline{c}^2}(\underline{K} - \underline{uc}, -\underline{ku} + \underline{c}, \underline{k})$, condition (10.16) is straightforward to check.*

Let us study a few pairs (g_1, g_2). In general, function g_k concerns a physical variable belonging for instance to $\{p, \rho, T, h, e, u, \rho u\}$.

Remark 20 *According to the equation of state, giving two independent thermodynamic variables give all the others.*

Hence there are essentially two cases :

- i) g_1 and g_2 consist in prescribing two thermodynamic variables,
- g_1 prescribes a thermodynamic variable and g_2 imposes a function of velocity (combination of velocity and a thermodynamic variable). For instance :

  ii1) g_2 imposes the velocity : $g_2(v) = u - u_{in}$,

  ii2) g_2 imposes the mass flow : $g_2(v) = v_2 - (\rho u)_{in}$,

  ii3) g_2 imposes the total enthalpy ($H = h + \frac{1}{2}u^2$) : $g_2(v) = H - H_{in}$.

10.1.1 Prescribing two thermodynamic variables at subsonic inlet

Let us first consider the case where two thermodynamic variables are imposed. According to the remark 20, we can consider that ρ, the density, and e, the internal energy, are given. Using the relations (6.10) and (6.11), we obtain :

$$d\rho = dv_1\,, \tag{10.17}$$

$$de = -\frac{e - \frac{1}{2}u^2}{\rho}dv_1 - \frac{u}{\rho}dv_2 + \frac{1}{\rho}dv_3\,, \tag{10.18}$$

so that

$$d\rho \wedge de = \frac{1}{\rho}(0, -1, -u)\,. \tag{10.19}$$

Hence it follows that condition (10.16) reads :

$$l_3 \cdot (d\rho \wedge de) = \frac{-1}{2\underline{\rho c}} \neq 0\,, \tag{10.20}$$

and we have shown the following result.

Proposition 4 *In the case of a subsonic inlet, all pair of independent thermodynamic variables is suitable.*

10.1.2 Prescribing a thermodynamic variable and velocity at subsonic inlet

Let us now turn ourselves towards the case the velocity is given : $g_2(v) = u - u_{in}$. If we use the fact that $u = \frac{v_2}{v_1}$ then we have

$$du = -\frac{u}{\rho}dv_1 + \frac{1}{\rho}dv_2\,. \tag{10.21}$$

Condition (10.16) reads then

$$dg_1(\underline{v}) \cdot (1, \underline{u}, \underline{H} - \frac{\underline{c}^2}{\underline{k}}) \neq 0\,, \tag{10.22}$$

or (see definition (10.8) of $r_2(v)$)

$$r_2(\underline{v}) \cdot dg_1(\underline{v}) \neq 0\,. \tag{10.23}$$

In the following part we restrict the study to three types of function g_1 :

- We assume that the pressure is given, *i.e.* $g_1(v) = p - p_{in}$. Since thermodynamic relations (6.16) and (6.17) imply that
$$dp = (c^2 - \frac{kp}{\rho})d\rho + k\rho de\,, \tag{10.24}$$
or similarly
$$dp = k\left\{(u^2 + \frac{c^2}{k} - H)dv_1 - udv_2 + dv_3\right\}\,, \tag{10.25}$$
it is easy to see that $g_1(v) = p - p_{in}$ does not satisfy (10.22).
- We assume that the internal energy is given, *i.e.* $g_1(v) = e - e_{in}$. Therefore from (10.18)
$$de = -(e - \frac{1}{2}u^2)dv_1 - udv_2 + dv_3\,, \tag{10.26}$$
and condition (10.22) implies that
$$\frac{\underline{p}}{\underline{\rho}} - \frac{\underline{c}^2}{\underline{k}} \neq 0\,. \tag{10.27}$$
For a perfect gas with constant specific heat coefficients, since $e = C_vT$, prescribing the internal energy is equivalent to prescribing temperature. Now let us remark that a polytropic gas for which $p = (\gamma-1)\rho e$ and $k = \gamma - 1$, satisfies the condition (10.27) since $-\frac{\underline{c}^2}{\gamma(\gamma-1)} \neq 0$.
- If the specific enthalpy is given, *i.e.* $g_1(v) = h - h_{in}$, since ($h = e + \frac{p}{\rho}$)
$$dh = \frac{1}{\rho}\left\{(c^2 - (k+1)H + (k+1)u^2)dv_1 - (k+1)udv_2 + (k+1)dv_3\right\}\,, \tag{10.28}$$
condition (10.22) reads $\underline{c}^2 \neq 0$ which is obviously satisfied.

Proposition 5 *In the case of a subsonic inlet, if we prescribe u, we can prescribe another condition $g_1(v) = 0$ if and only if (10.22) is satisfied. A condition which does not allow to impose the pressure but allow to impose the internal energy or the specific enthalpy.*

10.1.3 Prescribing a thermodynamic variable and mass flow at subsonic inlet

We study the case where one prescribes the mass flow and another condition. This case is also relevant from the physical point of view. Here we have $g_2(v) = v_2 - (\rho u)_{in}$ so that $dg_2 = dv_2$ and condition (10.16) reads
$$dg_1(\underline{v}) \cdot (\underline{k}, 0, \underline{uc} - \underline{K}) \neq 0\,. \tag{10.29}$$
If we take

- $g_1(v) = p - p_{in}$, then it leads to the condition $\underline{uc} \neq 0$ which is always satisfied since at an inlet we take $(\rho u)_{in} < 0$,
- $g_1(v) = e - e_{in}$, then condition (10.22) implies that $\underline{uc} - \underline{c}^2 + \frac{\underline{kp}}{\underline{\rho}} \neq 0$. A polytropic gas for which $p = (\gamma - 1)\rho e$ and $k = \gamma - 1$, satisfies this condition since $\underline{u} - \frac{\underline{c}}{\gamma} < 0$.

Proposition 6 *In the case of a subsonic inlet, if we prescribe the mass flow ρu, we can prescribe another condition $g_1(v) = 0$ if and only if (10.29) is satisfied. This condition allows to impose for instance the pressure.*

10.1.4 Prescribing a thermodynamic variable and total enthalpy at subsonic inlet

We assume here that total enthalpy is prescribed *i.e.* $g_2(v) = H - H_{in}$. Since by definition $H = e + \frac{p}{\rho} + \frac{1}{2}u^2$, we have :
$$dH = \frac{ku^2 + c^2 - (k+1)H}{\rho}dv_1 - \frac{ku}{\rho}dv_2 + \frac{k+1}{\rho}dv_3\,. \tag{10.30}$$
Then condition (10.16) reads in this case :
$$dg_1(\underline{v}) \cdot \left(\underline{k}(\underline{u} - \underline{c}) - \underline{c}, (\underline{c} - \underline{ku})(\underline{c} - \underline{u}), \underline{kH}(\underline{u} - \underline{c}) + \underline{c}(\underline{c}^2 - \underline{H})\right) \neq 0\,. \tag{10.31}$$
If for instance we impose entropy *i.e.* if we take $g_1(v) = s - s_{in}$, then since
$$ds = \frac{1}{\rho T}\left((u^2 - H)dv_1 - udv_2 + dv_3\right) \tag{10.32}$$
condition (10.31) reads $\underline{c} - \underline{u} \neq 0$. A condition obviously satisfied.

Proposition 7 *In the case of a subsonic inlet, if we prescribe the total enthalpy H, we can prescribe another condition $g_1(v) = 0$ if and only if (10.31) is satisfied. This condition allows to impose the entropy.*

10.1.5 The "incomplete" Riemann problem method

The method consists in finding μ, an intermediate state, such that

(a) $\underline{v}$ is connected to μ through a 1-simple wave and

(b) μ is connected to v through a 2-simple wave.

For $p = 1$ or 2, let us denote by W_k^p, $k = 1$ or 2, two independent p-pseudo Riemann invariants. It can be shown[6] that (a) is equivalent to

$$W_1^1(\underline{v}) = W_1^1(\mu)\,, \quad W_2^1(\underline{v}) = W_2^1(\mu)\,, \tag{10.33}$$

while (b) is equivalent to

$$W_1^2(\mu) = W_1^2(v)\,, \quad W_2^2(\mu) = W_2^2(v)\,. \tag{10.34}$$

In this method we have two vectors v and μ which are unknown, that is 6 scalar unknowns, and 6 scalar equations : (10.13)-(10.33)-(10.34). This must be compared with our method which involves only 3 scalar unknowns and equations, and moreover which does not impose to find and compute pseudo Riemann invariants.

We are going to compare the two approaches in more details and for that purpose, we begin by discussing the construction of pseudo Riemann invariants for the one dimensional Euler equations. For these equations there are at least two set of dependent variables which simplify the convection matrix $A(v)$. The first one are the variables (ρ, u, s) and the second one are (ρ, u, p). In the first variables, the right eigenvectors can be taken as :

variables (ρ, u, s)

$$\tilde{r}_1(v) = (\rho, -c, 0), \tag{10.35}$$

$$\tilde{r}_2(v) = \left(\left(\frac{\partial p}{\partial s}\right)_\rho, 0, -\left(\frac{\partial p}{\partial \rho}\right)_s\right), \tag{10.36}$$

$$\tilde{r}_3(v) = (\rho, c, 0). \tag{10.37}$$

While in the second set of variables, they can be taken as :

variables (ρ, u, p)

$$\tilde{r}_1(v) = (1, -\frac{c}{\rho}, c^2), \tag{10.38}$$

$$\tilde{r}_2(v) = (1, 0, 0), \tag{10.39}$$

$$\tilde{r}_3(v) = (1, \frac{c}{\rho}, c^2). \tag{10.40}$$

- *Computation of* 1*-Riemann invariants.* In view of the 3 different 1-right eigenvectors (10.7), (10.35) and (10.38), it is more convenient to work with the variables (ρ, u, s) since (3.31) simply reads :

$$\rho\frac{\partial W}{\partial \rho} - c\frac{\partial W}{\partial u} = 0\,. \tag{10.41}$$

  The third variable, namely s is clearly a 1-pseudo Riemann invariant[7]. According to the theory, we have another 1-pseudo Riemann invariant which is is functionally independent of s. If we look for a solution of the form $W = u + f(\rho, s)$, we automatically obtain that

$$W^1 = u + \int_{\rho_0}^{\rho} \frac{c(r,s)}{r} dr\,, \tag{10.42}$$

  is a 1-Riemann Invariant.

- *Computation of* 2*-Riemann invariants.* In view of the 3 different 2-right eigenvectors (10.8), (10.36) and (10.39), it is more convenient to work with the variables (ρ, u, p) since (3.31) simply reads :

$$\frac{\partial W}{\partial \rho} = 0\,. \tag{10.43}$$

  Here two independent 2-pseudo Riemann invariants are obviously u and p.

[6] In the regular case in the sense of Lax, see *e.g.* Smoller [51].

[7] In this case it is also a Riemann invariant.

- *Computation of* 3*-Riemann invariants.* In view of the 3 different 3-right eigenvectors (10.9), (10.37) and (10.40), it is more convenient to work with the variables (ρ, u, s) since (3.31) simply reads :

$$\frac{\partial W}{\partial \rho} + c\frac{\partial W}{\partial u} = 0\,. \tag{10.44}$$

Again the third variable, namely s, is clearly a 3-pseudo Riemann invariant and like for the 1-pseudo Riemann invariant case, we obtain that

$$W^3 = u - \int_{\rho_0}^{\rho} \frac{c(r,s)}{r} dr\,, \tag{10.45}$$

is a 3-Riemann Invariant.

Remark 21 *In the case of a perfect polytropic gas ($p = (\gamma-1)\rho e$) the 1-Riemann invariant (10.42) simplifies to $W^1 = u + \frac{2}{\gamma-1}c$ while the 3-Riemann invariant (10.45) simplifies to $W^1 = u - \frac{2}{\gamma-1}c$.*

We sum up these computations in Table 1.

	$k=1$	$k=2$	$k=3$
λ_k	$u-c$	u	$u+c$
k-pseudo Riemann invariants	$s,\ u + \int_{\rho_0}^{\rho} \frac{c(r,s)}{r} dr$	$u,\ p$	$s,\ u - \int_{\rho_0}^{\rho} \frac{c(r,s)}{r} dr$
k-pseudo Riemann invariants perfect polytropic gas	$s,\ u + \frac{2}{\gamma-1}c$	$u,\ p$	$s,\ u - \frac{2}{\gamma-1}c$

Table 1: k-pseudo Riemann invariant for one dimensional Euler equations

Let us now return to the use of the "incomplete" Riemann problem for the determination of the boundary state v. The internal state $\underline{v}$ is given and we have to find the two states v and μ satisfying the 6 scalar equations (10.13)-(10.33)-(10.34). Since we have explicit values for the pseudo Riemann invariants, we can rewrite the two last equations as :

$$s = s_1 \quad u - u_1 = \int_{\rho}^{\rho_1} \frac{c(r,s)}{r} dr \tag{10.46}$$

$$u_1 = \underline{u}, \quad p_1 = \underline{p}\,, \tag{10.47}$$

where the state v corresponds to (ρ, u, p), the state μ to (ρ_1, u_1, p_1) and $\underline{v}$ corresponds to $(\underline{\rho}, \underline{u}, \underline{p})$. These 4 nonlinear equations must be coupled with the given boundary conditions (10.13). This method appears to be much more complex than the one we propose, since we only have to add to (10.13) one equation which is here $l_3(\underline{v}) \cdot f(v) = l_3(\underline{v}) \cdot f(\underline{v})$ or explicitly :

$$(\underline{K} - \underline{uc})\rho u + (\underline{c} - \underline{ku})(\rho u^2 + p) + \underline{k}\rho u H = \underline{pc} + \underline{c}^2\underline{\rho u} - \underline{kup}\,. \tag{10.48}$$

On the other hand we have compared numerically the solutions on test cases and it appears that they give almost the same solution, see Section 9.

10.1.6 Conclusion

For a subsonic inlet boundary condition, even in the simplest cases of one dimensional Euler equation, the method of this paper leads to a simpler and faster algorithm for imposing the physical boundary conditions.

10.2 The case of a subsonic outlet boundary condition

We assume again that the problem is posed on the interval $]0, L[$ and now the outlet boundary is held at $x = L$ in order that x in equation (10.4) coincides with the coordinate along the outer normal. We have $0 < u < c$ and therefore with the notations of Section 3.1.3, $\chi = 1$ since there is one negative eigenvalue : $u - c$. Hence we have to prescribe one boundary condition that we write

$$g_1(v) = 0\,. \tag{10.49}$$

Condition (3.15) reads here

$$\frac{\partial g_1}{\partial v_1}(\underline{v}) + (\underline{u} - \underline{c})\frac{\partial g_1}{\partial v_2}(\underline{v}) + (\underline{H} - \underline{uc})\frac{\partial g_1}{\partial v_3}(\underline{v}) \neq 0\,.$$

Proposition 8 *The physical boundary conditions $g_1(v) = 0$ can be prescribed at a subsonic outlet if the condition*

$$\frac{\partial g_1}{\partial v_1}(\underline{v}) + (\underline{u} - \underline{c})\frac{\partial g_1}{\partial v_2}(\underline{v}) + (\underline{H} - \underline{uc})\frac{\partial g_1}{\partial v_3}(\underline{v}) \neq 0\,, \tag{10.50}$$

is satisfied.

Usually at a subsonic outlet it is standard to impose the pressure, that is to take $g(v) = p - p_{OUT}$ where p_{OUT} is a given pressure. According to Section 6.2.1 with $nd = 1$, we find that the left hand side of (10.50) is $\underline{c}^2$, a positive number. One can also wish to impose another quantity like velocity, mass flux, temperature,... Denoting by Δ the right hand side of (10.50), we obtain after some tedious computations the following results gathered in Table 2. These show that indeed it is possible to impose either the pressure or the velocity or else the temperature. It is not possible to prescribe the entropy (s) but it is also possible to impose the mass flux (ρu) or the total enthalpy (H). However in these last cases, an instability may occur if the flow becomes critical (sonic point) at the outlet.

Variable	$g_1(v)$	Δ
Pressure	$p - p_{OUT}$	$\underline{c}^2$
Velocity	$u - u_{OUT}$	$-\frac{\underline{c}}{\underline{\rho}}$
Mass flux	$\rho u - (\rho u)_{OUT}$	$\underline{u} - \underline{c}$
Temperature (perfect gas)	$T - T_{OUT}$	$\frac{2\underline{c}^2 + 3\gamma \underline{u}^2}{6\gamma C_v \underline{\rho}}$
Total Enthalpy	$H - H_{OUT}$	$\frac{\underline{c}(\underline{c} - \underline{u})}{\underline{\rho}}$
Entropy	$s - s_{OUT}$	0

Table 2: Values of Δ, the left hand side of (10.50)

10.3 The case of a wall boundary condition

We have already treated this case in the general case (Section 6.1). We assume that the wall is located at $x = L$. Recall that since $u = 0$ on the wall, the flux we are looking for simply reads as $f(v) = (0, p, 0)$ and therefore is totally determined by p. The characteristic analysis leads here to the equation :

$$l_3(\underline{v}) \cdot f(v) = l_3(\underline{v}) \cdot f(\underline{v})\,, \tag{10.51}$$

hence we obtain :

$$p = \frac{l_3(\underline{v}) \cdot f(\underline{v})}{l_3(\underline{v}) \cdot (0, 1, 0)}\,, \tag{10.52}$$

that is (compare with (6.4)) :

$$p = \underline{p} + \frac{\underline{\rho}\underline{c}^2 \underline{u}}{\underline{c} - \underline{k}\underline{u}}\,. \tag{10.53}$$

Remark 22 *For a polytropic gas, for which the equation of state is $p = (\gamma - 1)\rho e$ where $\gamma > 1$ is a given constant, we have $k = \gamma - 1$ and $c^2 = \frac{\gamma p}{\rho}$. Then formula (10.53) reads as*

$$p = \underline{p}\left(1 + \frac{\gamma \underline{u}}{\underline{c} - (\gamma - 1)\underline{u}}\right) = \underline{p}\left(1 + \frac{\gamma \underline{\mathcal{M}}}{1 - (\gamma - 1)\underline{\mathcal{M}}}\right)\,. \tag{10.54}$$

10.4 How to handle change of type

Let us discuss a typical case which arises in practice. Assume that we aim to capture a stationary state such that the right boundary $x = L$ corresponds to a subsonic outlet. The physical boundary condition in this case might be *e.g.* a prescribed pressure p_{OUT} as already discussed in Section 10.2. This situation assumes that at the boundary the fluid is indeed going outside of the domain and more precisely that $0 < u < c$. But since this stationary flow is computed by using the evolution equation, nothing guaranties that during the transient the fluid might temporarily enters into the domain *i.e.* that the velocity at the boundary (and/or at the cell adjacent to the boundary) might become negative $u < 0 < c$. In such a case the outlet becomes an inlet and according to the theory one must prescribe two physical boundary conditions. We have already one physical boundary condition, namely an imposed pressure, it is therefore natural to add another one to this condition. According to Section 10.1, we have various choices. The more natural one is to impose another thermodynamic

variable (Proposition 4). All the choices are equivalent in theory since once the pressure is given, any of the other thermodynamic variables (density, pressure, energy, enthalpy and entropy) will determine uniquely the thermodynamical state. Let us for instance assume that we decide to impose the density (it is the more natural choice since it is one of our computational variable v_1) $\rho = \rho_{OUT}$.
The structure of the algorithm is then as follows.

- If $0 < \underline{u} < \underline{c}$ then the flux $f(v) = (\rho u, \rho u^2 + p, \rho u H)$ is determined by solving the system :

$$\begin{aligned} p &= p_{OUT}\,, \\ l_2(\underline{v}) \cdot f(v) &= l_2(\underline{v}) \cdot f(\underline{v})\,, \\ l_3(\underline{v}) \cdot f(v) &= l_3(\underline{v}) \cdot f(\underline{v})\,. \end{aligned} \tag{10.55}$$

- If $\underline{u} = 0$ then $f(v) = (0, p_{OUT}, 0)$.
- If $-\underline{c} < \underline{u} < 0$ then the flux $f(v) = (\rho u, \rho u^2 + p, \rho u H)$ is determined by solving the system :

$$\begin{aligned} p &= p_{OUT}\,, \\ \rho &= \rho_{OUT} \\ l_3(\underline{v}) \cdot f(v) &= l_3(\underline{v}) \cdot f(\underline{v})\,. \end{aligned} \tag{10.56}$$

Remark 23 *We have not discussed yet the choice of ρ_{OUT}. Here again there are two possibilities. In the first one, ρ_{OUT} is prescribed because one has some insight on the value of ρ at the outlet for the stationary state under investigation. While in the second one, we simply take $\rho_{OUT} = \underline{\rho}$. This corresponds in a certain sense to the discretization of a Neumann's condition on ρ. In general this last condition is preferred since it does not produce an artificial boundary layer on ρ.*

11 Conclusions

In this paper we have presented a method for imposing boundary conditions for multidimensional hyperbolic systems of conservation laws. This method is general and very simple to write down and to implement. It does not rely on extra constructions like *e.g.* Riemann solvers or (pseudo) Riemann invariants. Moreover we have shown both via a mathematical proof and numerical evidence that it gives as good results as the more sophisticated method of partial Riemann problems, with the advantage that, besides being general, it is more efficient from the computational point of view.

12 Appendix : Construction of control volumes

To define the finite volume method, the computational domain Ω, taken to be a polygonal domain, has to be decomposed in small volumes K such that $\Omega = \cup_{K \in \mathcal{T}} K$. These small volumes K are polyhedra such that the boundary is the union of hypersurfaces $K \cap L$ where L belongs to the set $\mathcal{N}(K) = \{L \in \mathcal{T} / L \neq K$ and $K \cap L$ has positive $(nd-1)$-measure$\}$ and of hypersurfaces on the boundary of Ω *i.e.* $K \cap \partial\Omega$.
Here we assume that an unstructured mesh in the finite element sense composed of triangles in 2 dimensional space (for instance Figure 30) and of tetrahedra in 3 dimensional space is defined on the domain Ω. For defining control volumes on this given mesh, there are different choices.

12.1 The cell center finite volume approach

This method consists in approximating the average of the solution on the simplex itself (triangles in 2-D, tetrahedra in 3-D).

12.2 The vertex finite volume approach

The vertex finite volume approach, also called node-centered finite volume is very popular in the community of fluid dynamics (see Dervieux [14], Selmin [49], Stoufflet [48], Arminjon and Madrane [3], Carré and Dervieux [8], Debiez *et al* [13], Dervieux and Desideri [15], Eymard *et al* [21], Farhat and Lanteri [22], Feistauer *et al* [24],...). This method consists in approximating the average on the so-called dual mesh where the control volumes D are associated with vertices of the initial finite element mesh. For $nd = 2$, the volume D associated with vertex S is obtained by joining the center of gravity of elements K that contain the vertex S and the center of all faces of K sharing S. For vertices on the boundary, the volume is complete with the boundary

of the domain : see Figure 31. This dual mesh is also called "Voronoï" dual mesh. For $nd = 3$, the volume D associated with vertex S is the union of the subpolyhedrons resulting from the subdivision of polyhedrons K containing S by means of medians planes.

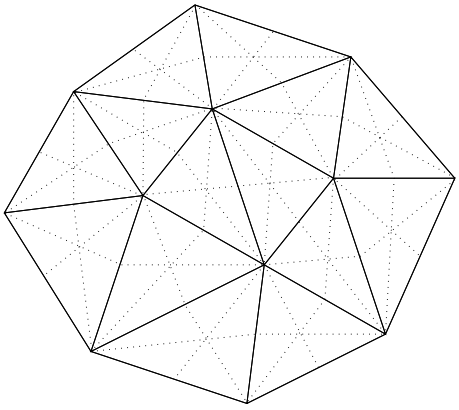

Figure 30: Cell F.V. : the control volumes are the simplex of the finite element mesh

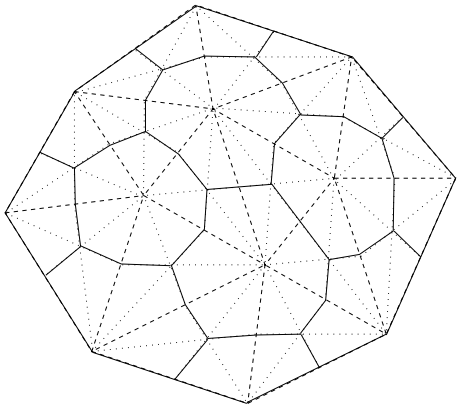

Figure 31: Vertex F.V. : Control volumes on a 2d triangular (dashed) mesh

Remark 24 *For $nd = 2$, if a linear approximation is used and if mass lumping or low order quadrature formulas are employed, one can prove that the scheme defined in Section 2.3.1 by the system of o.d.e.'s 2.64 and the flux 2.66 is equivalent to a conforming finite element method with an upwind treatment of the convective term (see for instance Idelsohn and Onate [33], Feistauer et al [24])...). An other advantage of this choice, concerns the discretization of second-order operator like viscous terms.*

Remark 25 *This method suffers from the building of the dual mesh. For instance, computations of the measure of volumes and the area of hypersurface are not an easy task in three dimensions. It may also involve much more numerical viscosity [41].*

Remark 26 *There is for $nd = 2$ and for a triangulation $\mathcal{T}$ made of triangles a slightly different method where control volumes (Figure 32) are obtained by joining the center of gravity of every triangles K surrounding the vertex S.*

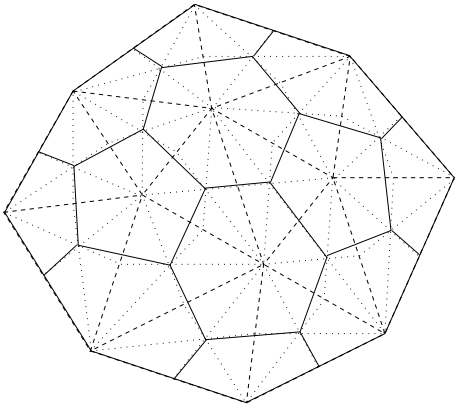

Figure 32: Modified vertex F.V. : Control volumes on a 2d triangular (dashed) mesh

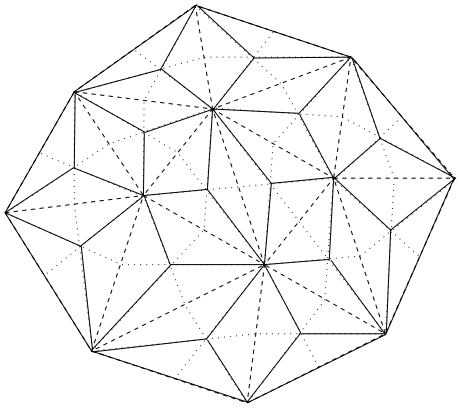

Figure 33: Barycentric F.V. : Control volumes on a 2d triangular (dashed) mesh

12.3 The barycentric finite volume approach

More recently, Dolejsi and Angot [16], Feistauer *et al* [23], Angot *et al* [2] and Dolejsi *et al* [17] have developed and used the barycentric finite volume where the average of solution is computed on control volumes that are defined by joining the barycenter of each volume of the primal mesh with its vertices : each control volume is then associated with a $nd - 1$ dimensional hypersurface of the initial mesh as it is illustrated on Figure 33.

13 References

[1] ANDERSON, J. *Computational Fluid Dynamics : the basics with applications*. Mc-Graw-Hill international editions, 1995.

[2] ANGOT, P., DOLEJSI, V., FEISTAUER, M., AND FELCMAN, J. Analysis of a combined barycentric finite volume—nonconforming finite element method for nonlinear convection-diffusion problems. *Appl. Math., Praha 43*, 4 (1998), 263–310.

[3] ARMINJON, P., AND MADRANE, A. A mixed finite volume/finite element method for 2-dimensional compressible navier-stokes equations on unstructured grids. In *Hyperbolic problems: Theory, numerics, applications. Proceedings of the 7th international conference, Zuerich, Switzerland* (February 1998), M. Fey and R. Jeltsch, Eds., vol. 129 of *Int. Ser. Numer. Math.*, Basel: Birkhaeuser, pp. 11–20.

[4] BARTH, T. J. Numerical methods for gasdynamic systems on unstructured meshes. In *An introduction to recent developments in theory and numerics for conservation laws (Freiburg/Littenweiler, 1997)*, vol. 5 of *Lect. Notes Comput. Sci. Eng.* Springer, Berlin, 1999, pp. 195–285.

[5] BESTION, D. The phase appearance and disappearance in the cathare code. In *Trends in Numerical and Physical Modeling for Industrial Multiphase Flows* (Cargese, 2000), http://www.cmla.ens-cachan.fr/Utilisateurs/perfortmans/Cargese00/.

[6] BOURE, J., AND DELHAYE, J.-M. General equations and two-phase flow modeling. In *Handbook of Multiphase Systems*, G. Hestroni, Ed. Hemisphere Publishing Corporation, 1982.

[7] BUFFARD, T., GALLOUËT, T., AND HÉRARD, J.-M. A sequel to a rough godunov scheme: Application to real gases. *Comput. Fluids 29*, 7 (2000), 813–847.

[8] CARRÉ, G., AND DERVIEUX, A. On the application of FMG to variational approximation of flow problems. *Int. J. Comput. Fluid Dyn. 12*, 2 (1999), 99–117.

[9] CHOI, H., AND MERKLE, C. The application of preconditioning the viscous flows. *J. Comp. Phys. 105* (1993), 207–223.

[10] CORTES, J. On the construction of upwind schemes for non-equilibrium transient two-phase flows. *Computers & Fluids 31* (2002), 159–182.

[11] DAFERMOS, C. M. *Hyperbolic conservation laws in continuum physics*. Springer-Verlag, Berlin, 2000.

[12] DARMOFAL, D. L., MOINIER, P., AND GILES, M. B. Eigenmode analysis of boundary conditions for the one-dimensional preconditioned Euler equations. *J. Comput. Phys. 160*, 1 (2000), 369–384.

[13] DEBIEZ, C., DERVIEUX, A., MER, K., AND NKONGA, B. Computation of unsteady flows with mixed finite volume/finite element upwind methods. *Internat. J. Numer. Methods Fluids 27*, 1-4, Special Issue (1998), 193–206. Finite elements in fluids.

[14] DERVIEUX, A. Steady euler simulations using unstructured meshes. In *Partial differential equations of hyperbolic type and applications.*, G. Geymonat, Ed. World Scientific Publishing Co.,John Wiley & Sons Ltd, 1987, pp. 33–111.

[15] DERVIEUX, A., AND DESIDERI, J.-A. Compressible flow solvers using unstructured grids. Tech. Rep. RR-1732, Rapport Institut National de Recherche en Informatique et en Automatique (INRIA), 1992.

[16] DOLEJSI, V., AND ANGOT, P. Finite volume methods on unstructured meshes for compressible flows. In *Finite volumes for complex applications* (Rouen, 1996), F. benkhaldoum and R. Vilsmeier, Eds., Edition Hermes, Rouen.

[17] DOLEJSI, V., FEISTAUER, M., FELCMAN, J., AND KLIKOVA, A. Error estimates for barycentric finite volumes combined with non conforming finite elements applied to nonlinear convection-diffusion problems. *submitted* (2000).

[18] DREW, D., AND LAHEY, R. Application of general constitutive principles to the derivation of multidimensional two-phase flow equations. *nt. J. Multiphase Flow 5* (1979), 243–264.

[19] DUBOIS, F. Partial riemann problem, boundary conditions and gas dynamics. In *Absorbing Boundaries and Layers, Domain Decomposition Methods. Applications to Large Scale Computations*, L. Tourrette and L. Halpern, Eds. Nova Science Publishers, Inc., New York, 2001, pp. 16–77.

[20] DUBOIS, F., AND LEFLOCH, P. Boundary conditions for nonlinear hyperbolic systems of conservation laws. *J. Differential Equations 71*, 1 (1988), 93–122.

[21] EYMARD, R., GALLOUËT, T., AND HERBIN, R. *Finite volume methods*, ciarlet, p.-a. and lions, j.-l. ed., vol. 7 of *Handbook of Numerical Analysis*. North-Holland, 2000, pp. 713–1020.

[22] FARHAT, C., AND LANTERI, S. Simulation of compressible viscous flows on a variety of mpps: Computational algorithms for unstructed dynamic meshes and performance results. *Comput. Methods Appl. Mech. Engrg. 119*, 1-2 (1994), 35–60.

[23] FEISTAUER, M., FELCMAN, J., AND DOLEJSI, V. Numerical simulation of compressible viscous flow through cascades of profiles. *ZAMM Journal of Applied Math. and Mech. 76*, Suppl. 4 (1996), 297–300.

[24] FEISTAUER, M., FELCMAN, J., AND LUKACOVA-MEDVIDÓVA, M. Combined finite element-finite volume solution of compressible flow. *J. Comput. and Applied Math. 63* (1995), 179–199.

[25] GHIDAGLIA, J.-M. Flux schemes for solving nonlinear systems of conservation laws. In *Innovative Methods for Numerical Solution of Partial Differential Equations* (2001), J.-J. Chattot and M. Hafez, Eds., World Scientific, Singapore.

[26] GHIDAGLIA, J.-M., KUMBARO, A., AND LE COQ, G. Une méthode volumes finis à flux caractéristiques pour la résolution numérique de lois de conservation. *C. R. Acad. Sci. Paris Série I 332* (1996), 981–988.

[27] GHIDAGLIA, J.-M., KUMBARO, A., AND LE COQ, G. On the numerical solution to two fluid models via a cell centered finite volume method. *Europ. Journal Mech. B, Fluids 20*, 6 (2001), 841–867.

[28] GODLEWSKI, E., AND RAVIART, P. *Numerical approximation of Hyperbolic Systems of Conservation Laws*. Springer Verlag, New-York, 1996.

[29] GODUNOV, S. Finite difference method for numerical method for numerical computations of discontinuous solutions of the equations of fluid dynamics. *Mat. Sbornik*, 47 (1959), 271–306.

[30] HECHT, F., AND SALTEL, E. *Un Editeur de Maillage et de Contours en 2 dimensions*. Rapport Institut National de Recherche en Informatique et en Automatique (INRIA), 1998.

[31] HIRSCH, C. *Numerical computation of internal and external flows. Volume 1: Fundamentals of numerical discretization*. Wiley Series in Numerical Methods in Engineering. Wiley-Interscience Publication., 1988.

[32] HIRSCH, C. *Numerical computation of internal and external flows. Volume 2: Computational methods for inviscid and viscous flows*. Wiley Series in Numerical Methods in Engineering. Wiley-Interscience Publication., 1990.

[33] IDELSOHN, S., AND ONATE, E. Finite volumes and finite elements: Two "good friends". *Int. J. Numer. Methods Eng. 37*, 19 (1994), 3323–3341.

[34] ISHII, M. *Thermo-Fluid Dynamic Theory of Two-Phase Flow*. Eyrolles, Paris, 1975.

[35] JOHN, F. Finite amplitude waves in a homogeneous isotropic elastic solid. *Commun. Pure Appl. Math. 30* (1977), 421–446.

[36] KREISS, H.-O., AND BUSENHART, H. U. *Time-dependent partial differential equations and their numerical solution*. Birkhäuser. viii, 82 p., 2001.

[37] LEE, D. *Local Preconditioning of the Euler and Navier-Stokes equations*. PhD thesis, University of Michigan, 1994.

[38] LIONS, J., AND MAGENES, E. *Problemes aux limites non homogenes et applications. Vol. 1, 2*. Dunod 1: XIX, 372 p.; 2: XV, 251 p., Paris, 1968.

[39] LIONS, J., AND MAGENES, E. *Non-homogeneous boundary value problems and applications. Vol. I. Translated from the French by P. Kenneth.* Die Grundlehren der mathematischen Wissenschaften. Band 181. Springer-Verlag. XVI,357 p., Berlin-Heidelberg-New York, 1972.

[40] MARSDEN, J. E., AND HUGHES, T. J. R. *Mathematical foundations of elasticity*. Dover Publications Inc., New York, 1994. Corrected reprint of the 1983 original.

[41] PASCAL, F. FV method with characteristic flux : influence of the geometrical aspect of control volumes. In *Finite volumes for complex applications* (2002), E. Hermes, Ed.

[42] POWELL, K. G. Collocated upwind schemes for ideal MHD. In *Innovative methods for numerical solutions of partial differential equations (Arcachon, 1998)*. World Sci. Publishing, River Edge, NJ, 2002, pp. 10–48.

[43] POWELL, K. G., ROE, P. L., LINDE, T. J., GOMBOSI, T. I., AND DE ZEEUW, D. L. A solution-adaptive upwind scheme for ideal magnetohydrodynamics. *J. Comput. Phys. 154*, 2 (1999), 284–309.

[44] RANSOM, V.-H. Faucet flow, oscillating manometer, and expulsion of steam by sub cooled water. In *Numerical Benchmark Tests, Multiphase Science and Technology*, G. Hewitt, J. Delhaye, and N. Zuber, Eds., vol. 3. Hemisphere Publishing Corporation, 1987.

[45] RIZZI, A., AND VIVIAND, H., Eds. *Numerical methods for the computation of inviscid transonic flows with shock waves*. Friedr. Vieweg & Sohn, Braunschweig, 1981. Papers from the GAMM Workshop.

[46] ROE, P. Approximate riemann solvers, parameter vectors, and difference schemes. *J. Comput. Phys.*, 43 (1981), 357–372.

[47] ROE, P. L., AND BALSARA, D. S. Notes on the eigensystem of magnetohydrodynamics. *SIAM J. Appl. Math. 56*, 1 (1996), 57–67.

[48] ROSTAND, P., AND STOUFFLET, B. TVD schemes to compute compressible viscous flows on unstructured meshes. In *Nonlinear hyperbolic equations—theory, computation methods, and applications (Aachen, 1988)* (Braunschweig, 1989), J. Ballmann and R. Jeltsch, Eds., vol. 24 of *Notes Numer. Fluid Mech.*, Vieweg, pp. 510–520.

[49] SELMIN, V. The node-centred finite volume approach: Bridge between finite differences and finite elements. *Comput. Methods Appl. Mech. Engrg. 102*, 1 (1993), 107–138.

[50] SERRE, D. *Systems of conservation laws. 1*. Cambridge University Press, Cambridge, 1999. Hyperbolicity, entropies, shock waves, Translated from the 1996 French original by I. N. Sneddon.

[51] SMOLLER, J. *Shock waves and reaction-diffusion equations*, second ed. Springer-Verlag, New York, 1994.

[52] STOER, J., AND BULIRSCH, R. *Introduction to numerical analysis*, second ed. Springer-Verlag, New York, 1993. Translated from the German by R. Bartels, W. Gautschi and C. Witzgall.

[53] TOUMI, I. An upwind numerical method for a six equation two-fluid model. *Nuclear Science and Engineering 123* (1996), 147–168.

[54] TOUMI, I., AND KUMBARO, A. An approximate linearized riemann solver for a two-fluid model. *J. Comput. Phys. 124*, 2 (1996), 286–300.

[55] TSYNKOV, S. V. Numerical solution of problems on unbounded domains. a review. *Appl. Numer. Math. 27*, 4 (1998), 465–532.

[56] TURKEL, E. Preconditioned methods for solving the incompressible and low speed compressible equations. *J. Comp. Phys. 72* (1987), 277–298.

[57] TURKEL, E. Review of preconditionning methods for fluid dynamics. *App. Num. Math. 12* (1993), 257–284.

[58] TURKEL, E., RADESPIEL, R., AND KROLL, N. Assessment of preconditioning methods for multidimensional aerodynamics. *Comput. Fluids 26*, 6 (1997), 613–634.

[59] VAN LEER, B., LEE, W., AND ROE, P. Characteristic time-stepping or local preconditioning of the euler equations. *AIAA Paper 91* (1991), 1552.

[60] WHITHAM, G. B. *Linear and nonlinear waves*. Wiley-Interscience [John Wiley & Sons], New York, 1974. Pure and Applied Mathematics.