FLUX SCHEMES FOR SOLVING NONLINEAR SYSTEMS OF CONSERVATION LAWS

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Abstract. We introduce a distinction among the classical three point finite difference schemes for nonlinear systems of conservation laws : those which are flux schemes and those which are not. We discuss at the continuous level why we think that these schemes are of interest. We also study some of their properties and their extensions to the multidimensional (finite volumes) case.

1 Introduction

Let us consider a system of $m$-conservation laws ($m \geq 1$)

$$\frac{\partial v}{\partial t} + \frac{\partial \varphi}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0.$$  

Here the flux $\varphi$ is related, at the continuous level, to the so-called conservative variable $v$ by a nonlinear function $f : G \rightarrow \mathbb{R}^m$ ($G$ is an open set in $\mathbb{R}^m$) :

$$\varphi = f(v).$$

In view of the discretization of (1) and given two positive real numbers $\Delta t$ and $\Delta x$, it is classical to associate to this system of equations a discrete dynamical system $(v^n_j)_{j \in \mathbb{Z}} \rightarrow (v^{n+1}_j)_{j \in \mathbb{Z}}$ where

$$v^{n+1}_j = v^n_j - \lambda(g^n_{j+\frac{1}{2}} - g^n_{j-\frac{1}{2}}),$$

and $\lambda = \Delta t/\Delta x$. The vector $g^n_{j+\frac{1}{2}}$ is the numerical flux, and for a three point explicit numerical scheme we have :
where the function \( g(\lambda; \cdot, \cdot) \) is to be specified.

Consistency, which is necessary if we want that solutions to (3) “converge” to solutions to (1), simply amounts imposing the condition

\[
g(\lambda; v, v) = f(v), \quad \forall v \in G, \tag{5}
\]

and we shall assume this identity from now on.

Let us for the sake of simplicity in the notations omit the dependence on \( \lambda \) in \( g \) and set also

\[
g(v, w) = \frac{f(v) + f(w)}{2} + h(v, w). \tag{6}
\]

so that (5) simply reads \( h(v, v) = 0 \). The function \( h \) represents the noncentered part in the numerical flux and as is well-known, it is the part that contributes to the stability of the scheme. Since \( h(v, v) = 0 \), we expect that \( h(v, w) \) is the product of a \( m \times m \) matrix \( M(v, w) \) by the vector \( w - v \). Indeed all the consistent numerical flux are of this form (see Proposition 1):

\[
h(v, w) = \frac{1}{2} M(v, w)(w - v). \tag{7}
\]

We would like in this article to consider a subclass of schemes that we term as flux schemes according to the following definition.

**Definition 1** The numerical flux \( g \) corresponds to a flux scheme when there exists a \( m \times m \) matrix \( S(v, w) \) such that

\[
h(v, w) = \frac{1}{2} S(v, w)(f(w) - f(v)). \tag{8}
\]

As a result, when (8) holds true, we observe that the numerical flux is a linear combination (whose coefficients depends on \( v \) and \( w \)) of \( f(v) \) and \( f(w) \):

\[
g(v, w) = \frac{(I - S(v, w))}{2} f(v) + \frac{(I + S(v, w))}{2} f(w). \tag{9}
\]

The goal of this article is to study some properties of schemes corresponding to such numerical fluxes.

**Remark 1** When we only have (7), we can write

\[
g(v, w) = \frac{f(v) + f(w)}{2} + \frac{1}{2} M(v, w)(w - v), \tag{10}
\]

and \( M \) is usually termed as the viscosity matrix. We shall therefore term the corresponding scheme as a viscous scheme.
2 Some remarks at the continuous level

Let us first consider a smooth solution to (1) - (2). We denote by \( A(v) \) the gradient of \( f(v) \):

\[
A_{ij}(v) = \frac{\partial f_i}{\partial v_j}(v), \quad 1 \leq i, j \leq m, \quad \forall v \in G. \tag{11}
\]

We multiply (1) by \( A(v) \) and observe that since \( A(v) \frac{\partial v}{\partial t} = \frac{\partial \varphi}{\partial t} \), we have

\[
\frac{\partial \varphi}{\partial t} + A(v) \frac{\partial \varphi}{\partial x} = 0. \tag{12}
\]

We deduce from this computation that, in a certain sense, the flux \( \varphi \) satisfies a linear equation (of which coefficients depend on \( v \)).

Let us now prove that equation (12) remains valid across a shock. We consider an open set \( U \subset \mathbb{R} \times \mathbb{R}_+^* \), which is divided in two open parts \( U^+ \) and \( U^- \) by a \( C^1 \) curve \( \Sigma = \{(x(t), t), a < t < b\} \). A function \( v \), which is smooth in \( U \setminus \Sigma \), will be a weak solution to (1)-(2) in \( U \), if and only if it is a classical solution in \( U \setminus \Sigma \) and accross \( \Sigma \), the Rankine-Hugoniot jump condition is fulfilled:

\[
[f(v)] = \sigma[v], \quad [w] \equiv w^+ - w^- \quad \text{and} \quad w^\pm = \lim_{x \to x(t)} \{w(x,t), (x,t) \in U^\pm\}. \tag{13}
\]

We extend \( \sigma \) to the whole line \( \{(y,t), y \in \mathbb{R}\} \) by setting

\[
\sigma(y,t) = \sigma(t), \quad \forall y \in \mathbb{R}. \tag{14}
\]

Since \([\varphi - \sigma v] = 0\), we can define across \( \Sigma \) the functions \( A(v) \frac{\partial (\varphi - \sigma v)}{\partial x} \) and \( (\varphi - \sigma v)_t \) as \( L^1_{\text{loc}} \)-functions and then consider

\[
\mathcal{N} \equiv (\varphi - \sigma v)_t + A(v)(\varphi - \sigma v)_x + \sigma_t v. \tag{15}
\]

In \( U^+ \) or \( U^- \), \( \mathcal{N} \) is equal to zero since (1) implies

\[
v_t + A(v)v_x = 0 \text{ in } U \setminus \Sigma. \tag{16}
\]

Since \( \mathcal{N} \) is a \( L^1_{\text{loc}} \)-function, we deduce that \( \mathcal{N} \) is almost everywhere equal to zero:

\[
(\varphi - \sigma v)_t + A(v)(\varphi - \sigma v)_x = -\sigma_tv. \tag{17}
\]

This gives a weak formulation to (12) across a shock.

**Theorem 1** Let \( v \) be a discontinuous solution to (1) - (2) with a finite number of shocks that do not interact. Then equation (12) holds true in the sense that \( \mathcal{N} \) defined by (15) is almost everywhere equal to zero.
This result is according to us a mathematical reason for which flux schemes are of interest. Indeed, equation (12) shows that a kind of “superposition principle” should be true at the discrete level and flux schemes, as already noticed (see (9)), are reminiscent of such a principle.

Remark 2 In the case of a shock interaction, it is not possible to give a sense to (17) since it may happen that \( \sigma \) undergoes a discontinuity at the point of interaction and then \( \sigma v \) cannot be defined.

3 A catalogue of numerical schemes

Let us consider a few numerical flux.

- We start with the Lax-Friedrichs scheme whose numerical flux reads as:

\[
g^{LF}(v, w) = \frac{f(v) + f(w)}{2} - \frac{1}{2\lambda}(w - v). \tag{18}
\]

- Next for the Lax-Wendroff scheme:

\[
g^{LW}(v, w) = \frac{f(v) + f(w)}{2} - \frac{\lambda}{2}A(v, w)(f(w) - f(v))
\]

where \( A \) maps \( G \times G \) into \( \mathcal{M}_m(\mathbb{R}) \) and satisfies

\[
A(v, w) = A(v). \tag{19}
\]

- Then for the Godunov scheme:

\[
g^{GOD}(v_l, v_r) = f(w_R(0, v_l, v_r)), \tag{20}
\]

where \( w_R(\xi; v_l, v_r) \) is the value at time \( t = 1 \) of the entropic solution (when it exists) to the Riemann problem (1) - (3) with initial data \( v(x, 0) = v_l \) for \( x < 0 \) and \( v(x, 0) = v_r \) for \( x > 0 \).

- In order to define Roe’s scheme, we assume that we can construct a mapping \( R \) from \( G \times G \) into \( \mathcal{M}_m(\mathbb{R}) \) such as:

\[
A_R(v, w) \text{ has real eigenvalues and a corresponding set of eigenvectors that are complete in } \mathbb{R}^m, \tag{21}
\]

\[
A_R(v, v) = A(v), \tag{22}
\]

\[
A_R(v, w)(w - v) = f(w) - f(v). \tag{23}
\]

Let us denote by \( sgn(M) \) the sign of a matrix \( M \). For example when \( M \) satisfies (21) we have:

\[
sgn(M) = P^{-1} diag(sgn(\lambda_i))P, \text{ where } M = P^{-1}diag(\lambda_i)P, \tag{24}
\]

and \( sgn(\lambda) = 0 \) if \( \lambda = 0 \) and \( \lambda/|\lambda| \) otherwise. We also set \( |M| = sgn(M)M. \)
Then Roe’s scheme corresponds to
\[ g_{ROE}(v, w) = f(v) + f(w) - \frac{1}{2} |A_R(v, w)|(w - v). \]  
(25)

And we observe that thanks to (23),
\[ g_{ROE}(v, w) = f(v) + f(w) - \frac{1}{2} \text{sgn}(A_R(v, w))(f(w) - f(v)). \]  
(26)

Remark 3 These two expressions show that Roe’s scheme is both a viscous scheme (25) and a flux scheme (26).

– Gallouet and Massella [3] have introduced a modification of Roe’s method by taking (they call this scheme VFRoe, for Finite Volume Roe)
\[ g_{VFRoe}(v, w) = f(v) + f(w) - \frac{1}{2} A(v + w - 2 v)(w - v). \]  
(27)

– Finally we end up with the scheme introduced in Ghidaglia, Kumbaro and Le Coq [4] (that we termed as the characteristic flux scheme). Let \( \mu(v, w) \) denote a mean value between \( v \) and \( w \) which is independent of the continuous flux e.g. \( \mu(v, w) = \frac{v + w}{2} \).

By using (12) we construct in [4] the following numerical flux :
\[ g_{CF}(v, w) = f(v) + f(w) - \frac{1}{2} \text{sgn}(A(\frac{v + w}{2}))(f(w) - f(v)). \]  
(28)

The motivation for the construction of such schemes comes from the case where we are considering complex systems (e.g. those occurring in two phase flows). In such a case Roe’s solver is not available, and this numerical flux has shown very good results in 1D, 2D and 3D computations (see e.g. Boucker [2] and Ghidaglia, Kumbaro, Le Coq [4]).

Remark 4 In the Characteristic Flux scheme (28), the matrix \( \text{sgn}(A) \) which appears produces the upwinding (as in Roe’s solver). In the context of two-phase flows, it is common that the convection operator that occurs is not hyperbolic. In such a case, the matrix \( A \) can have complex eigenvalues. In this case we take for \( \text{sgn}(M) \) the matrix given by (24) where this time \( \lambda_i \) denotes the real part of the eigenvalues. See also Halaoua [5] for an extension of Roe’s scheme in this context (complex eigenvalues).

On the relation between flux schemes and viscous schemes

Let us observe that every flux scheme can be written as a viscous scheme. Indeed, denoting by \( A_T(v, w) \) the Taylor matrix :
we see that any numerical flux corresponding to a flux scheme:

\[ g(v, w) = \frac{f(v) + f(w)}{2} + \frac{1}{2} S(v, w)(f(w) - f(v)), \]  

(30)
can be recast as a numerical flux corresponding to a viscous scheme:

\[ g(v, w) = \frac{f(v) + f(w)}{2} + \frac{1}{2} M(v, w)(w - v), \]  

(31)
with viscosity matrix

\[ M(v, w) = S(v, w) A_T(v, w). \]  

(32)

In general the converse is false. It is obvious that the Lax-Friedrichs scheme is not a flux scheme and we shall prove by a rather sophisticated, although interesting, way that Godunov’s scheme is not a flux scheme either (see Proposition 2).

**Proposition 1** Any consistent scheme can be written as a viscous scheme.

**Proof.** This follows from Taylor’s expansion formula:

\[ h(v, w) - h(v, v) = \left( \int_0^1 D^2 h(v, \theta v + (1 - \theta)w) d\theta \right) (w - v) \]  

(33)
and therefore we can take

\[ M(v, w) = 2 \int_0^1 D^2 h(v, \theta v + (1 - \theta)w) d\theta. \]  

(34)

Hence provided the numerical flux function \( g \) is \( C^1 \) (or \( W^{1,\infty} \)) we can write any three points scheme as a viscous scheme. However the expression of the associated viscosity matrix \( M(v, w) \) can be out of reach. For example, we do not know whether Godunov’s scheme has a “simple” viscosity matrix.

## 4 On the relation with source terms

In Alouges, Ghidaglia and Tajchman [1], we have considered the interaction between the numerical flux used and the discretisation of source terms. We consider instead of (1), the equation:

\[ \frac{\partial v}{\partial t} + \frac{\partial \varphi}{\partial x} = S, \quad x \in \mathbb{R}, \quad t > 0, \]  

(35)
where \( S \) depends only on \( x \). Now we discretize this equation by
\begin{align*}
  v_j^{n+1} &= v_j^n - \lambda (g_{j+\frac{1}{2}} - g_{j-\frac{1}{2}}) + \Delta t \Sigma_j^n, \\
  g_{j+\frac{1}{2}} &= g(\lambda; v_j^n, v_{j+1}^n),
\end{align*}

where

\begin{equation}
  g(v, w) = \frac{f(v) + f(w)}{2} - U(v, w) \frac{f(w) - f(v)}{2}.
\end{equation}

As discussed in [1], if we simply take:

\begin{equation}
  \Sigma_j^n = \Sigma_j = \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} S(x) dx,
\end{equation}

then due to the fact that the numerical flux is not centered (upwind bias) one can observe large errors on any steady solution under investigation. We have therefore introduced the notion of enhanced consistency as follows. Denoting by

\begin{equation}
  \phi = \phi^{\text{conv}}(x) \equiv \int_0^x S(y) dy,
\end{equation}

we say that $\Sigma_j^n$ satisfies, with respect to the numerical flux $g$, the enhanced consistency property when we have:

\begin{equation}
  f(v_j^n) = \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} \phi^{\text{conv}}(x) dx, \quad \forall j \in \mathbb{Z},
\end{equation}

then $v_j^{n+1}$ given by (36) must be equal to $v_j^n$. This condition can be equivalently formulated as

\begin{equation}
  \Sigma_j^n = \frac{f_{j+1/2}^n - f_{j-1/2}^n}{\Delta x_j},
\end{equation}

if $v_j^n$ satisfies (41).

With these notations, we show in [1] the following results.

**Theorem 2** Let $g$ be given in the form (38) and denote by $U_j^n = U(v_j^n, v_{j+1}^n)$. The enhanced consistency will be satisfied if we discretize the forcing term $S$ according to the following formula

\begin{equation}
  \Sigma_j^n = \frac{1}{2\Delta x_j} \left\{ \int_{x_{j-1}}^{x_{j+1}} S(y) dy - U_j^n \int_{x_j}^{x_{j+1}} S(y) dy + U_{j-1}^n \int_{x_{j-1}}^{x_j} S(y) dy \right\}.
\end{equation}

\begin{equation}
  \Sigma_j^n = \frac{I + U_{j-1}^n}{4} \Delta x_{j-1} S_{j-1} + \frac{I - U_{j-1}^n + U_j^n}{2} S_j + \frac{I - U_j^n}{4} \Delta x_{j+1} S_{j+1}.
\end{equation}

**Corollary 1** Assuming that $S$ is given by a piecewise constant function $S(x) = S_j$ for $x \in [x_{j-1/2}, x_{j+1/2}]$, the formula (43) reads

\begin{equation}
  \Sigma_j^n = \frac{I + U_{j-1}^n}{4} \Delta x_{j-1} S_{j-1} + \frac{I - U_{j-1}^n + U_j^n}{2} S_j + \frac{I - U_j^n}{4} \Delta x_{j+1} S_{j+1}.
\end{equation}
In order to illustrate this formula, let us show what it means in the case of a single linear equation $v_t + c v_x = S$ where e.g. $c > 0$ and $v \in \mathbb{R}$. Here the usual upwind scheme amounts to take $U^n_j = 1$, i.e. $g(v, w) = v$ and (44) reads as:

$$\Sigma^n_j = \frac{1}{2} \frac{\Delta x_{j-1}}{\Delta x_j} S_{j-1} + \frac{1}{2} S_j.$$  \hspace{1cm} (45)

5 Relations with entropies

At the level of a general system (1) - (2), there is no universal notion of entropy and the minimal requirement is that a good notion of entropy should select one solution to the Riemann problem which is moreover stable in a suitable sense.

For physical systems (i.e. systems steaming from problems in continuous physics) usually one has an entropy function i.e. a strictly convex function $\eta : G \rightarrow \mathbb{R}$, sufficiently smooth, to which one can associate an entropy flux $\psi : G \rightarrow \mathbb{R}$ such that smooth solutions to (1) - (3) satisfies the additional conservation law

$$\eta_t + \psi_x = 0,$$  \hspace{1cm} (46)

which amounts simply to the algebraic relations :

$$A(v) \nabla_x \eta(v) = \nabla_x \psi(v).$$  \hspace{1cm} (47)

Weak solutions to (1) - (2) will not satisfy (46) in general but

$$\eta_t + \psi_x \leq 0,$$  \hspace{1cm} (48)

in the sense of distributions.

If we consider a shock (see section 2) we know that (1) - (3) is equivalent to the Rankine-Hugoniot condition (13) :

$$[f(v)] = \sigma[v],$$  \hspace{1cm} (49)

while (48) amounts to

$$[\psi(v)] \leq \sigma[v],$$  \hspace{1cm} (50)

provided we take $U^+ = \{(y, t) \in U; y > x(t)\}$.

Let us now consider two states $v_l$ (left) and $v_r$ (right) (with $v_l \neq v_r$) such that there exists $\sigma \in \mathbb{R}$ for which

$$\varphi_l - \varphi_r = \sigma(v_l - v_r),$$  \hspace{1cm} (51)

where $\varphi_l \equiv f(v_l)$ and $\varphi_r \equiv f(v_r)$.

We choose the time step $\Delta t$ and the space step $\Delta x$ so that
\[ \sigma \frac{\Delta t}{\Delta x} = \sigma \lambda \in \mathbb{Z}. \]  

(52)

For example, when \( \sigma = 0 \) (a stationary shock), this condition holds for every \( \Delta t \) and \( \Delta x \).

Then considering any flux scheme

\[ v_{j}^{n+1} = v_{j}^{n} - \lambda (g_{j+\frac{1}{2}}^{n} - g_{j-\frac{1}{2}}^{n}), \quad g_{j+\frac{1}{2}}^{n} = g(\lambda; v_{j}^{n}, v_{j+1}^{n}), \]  

(53)

\[ g(v, w) = \frac{f(v) + f(w)}{2} + \frac{1}{2} S(v, w)(f(w) - f(v)); \]  

(54)

and starting with the initial data

\[ v_{j}^{0} = v_{l} \text{ for } j \leq 0 \text{ and } v_{j}^{0} = v_{r} \text{ for } j \geq 1; \]  

(55)

we generate a pure shock wave

\[ v_{j}^{n} = v_{j-(\sigma \lambda)n}, \quad \forall j \in \mathbb{Z}, \quad \forall n \in \mathbb{N}. \]  

(56)

But now since we can exchange the subscript left and right in (51), we can also propagate a pure shock wave connecting from left to right \( v_{r} \) and \( v_{l} \).

This shows immediately that flux-schemes are not entropic. Indeed, the entropy condition (50) reads here as

\[ \psi(v_{r}) - \psi(v_{l}) < \sigma(\eta(v_{r}) - \eta(v_{l})) \]  

(57)

since in general equality cannot hold.

Hence one of the two shock waves that we just constructed is not entropic (observe that as \( \Delta t \) and \( \Delta x \) goes to zero with \( \lambda \) constant, the discrete shock waves converge to continuous ones).

A by-product of this construction is the following result.

**Proposition 2** Godunov scheme is not a flux scheme.

This follows from the well-known fact (see e.g. [6]) that Godunov scheme is entropic.
6 Generalization to multidimensional schemes

The generalization of conservative finite differences in space dimension larger than 1 are the finite volume methods. Here again, numerical methods can be described by numerical fluxes. Let us discuss briefly this now. We only perform a space discretization of the operators so that we shall end up with a system of o.d.e.’s.

We consider a system of \( m \) balance equations :

\[
\frac{\partial v}{\partial t} + \nabla \cdot F(v) = 0 ,
\]

(58)

here \( \nabla \cdot F(v) = \sum_{j=1}^{nd} \frac{\partial F_j(v)}{\partial x_j} \), where \( F_j \) maps \( \mathbb{R}^m \) into \( \mathbb{R}^m \). This equation is posed in a \( nd \)-dimensional domain \( \Omega \) (\( nd = 1, 2 \) or 3 in practice).

We assume that the computational domain \( \Omega \) is decomposed in smaller volumes (the so-called control volumes) \( K : \Omega = \bigcup_{K \in \mathcal{T}} K \) and we assume for the sake of simplicity in the exposition that \( \Omega = \bigcup_{K \in \mathcal{T}} K \) is ”conformal” i.e. that it is a finite element triangulation of \( \Omega \). In practice one can use triangles for \( nd = 2 \) and tetrahedrons for \( nd = 3 \). The cell-center finite volume approach for solving (58) consists in approximating the means

\[
v_K(t) = \frac{1}{\text{vol}(K)} \int_K v(x,t) \, dx ,
\]

(59)

where \( \text{vol}(K) \) denotes the \( nd \)-dimensional volume of \( K \) and \( \text{area}(A) \) stands for the \( (nd-1) \)-dimensional volume of an hypersurface \( A \). Integrating (58) on \( K \) makes the normal fluxes, \( F_{\partial K} \), appear

\[
F_{\partial K}^\nu(t) = \int_{\partial K} F(v(\sigma,t)) \nu(\sigma) \, d\sigma ,
\]

(60)

where \( \partial K \) is the boundary of \( K \), \( \nu(\sigma) \) the unit external normal on \( \partial K \) and \( d\sigma \) denotes the \( (nd-1) \)-volume element on this hypersurface. Indeed, we have according to (58) :

\[
\frac{dv_K}{dt} + \frac{1}{\text{vol}(K)} F_{\partial K}^\nu = 0 .
\]

(61)

The heart of the matter in finite volume methods consists in providing a formula for the normal fluxes \( F_{\partial K}^\nu \) in terms of the \( \{v_L\}_{L \in \mathcal{T}} \). Assuming that the control volumes \( K \) are polyhedra, as is most often the case, the boundary \( \partial K \) is the union of hypersurfaces \( K \cap L \) where \( L \) belongs to the set \( \mathcal{N}(K) \), the set of \( L \in \mathcal{T}, L \neq K \), such that \( K \cap L \) has positive \( (nd-1) \)-measure. We can therefore decompose the normal flux as a sum :

\[
F_{\partial K}^\nu = \sum_{L \in \mathcal{N}(K)} F_{K,L} ,
\]

(62)

where \( \nu_{K,L} \) points into \( L \) :
Motivated by the case where (58) describes wave propagation phenomena, it is natural to look for an approximation of (63) in terms of $v_K(t)$ and $v_L(t)$:

$$F_{K,L} \approx \text{area}(K \cap L) \Phi(v_K, v_L; K, L),$$

where $\Phi$ is the numerical flux to be described.

At this level of generality $\Phi$ must satisfy 2 structural properties:

(i) consistency: $\Phi(w, w; K, L) = F(w) \cdot \nu_{K,L},$

(ii) conservation: $\Phi(w, v; K, L) = -\Phi(w, v; L, K)$.

In this case we introduce the following definition of flux schemes.

**Definition 2** The numerical flux $\Phi$ corresponds to a flux scheme when there exists a matrix $U(v; w; K, L)$ such that

$$\Phi(v, w; K, L) = \frac{F(v) + F(w)}{2} \cdot \nu_{K,L} - U(v, w; K, L) \cdot \frac{F(w) - F(v)}{2} \cdot \nu_{K,L}. \quad (65)$$

Then any of the numerical flux schemes introduced in the 1D case can be generalized by taking for the matrix $U(v; w; K, L)$ the matrix which corresponds to $f(v) \equiv F(v) \cdot \nu_{K,L}$. For example the extension of formula (28) to the multidimensional case is given thought the following definition.

**Definition 3** The numerical flux of the "VFFC" method is obtained by formula (65) when we take:

$$U(v, w; K, L) = \text{sgn}(\frac{\partial F(v) \cdot \nu_{K,L}}{\partial v}|_{v=\mu(v, w; K, L)}), \quad (66)$$

where $\mu(v, w; K, L)$ is a mean between $v_K$ and $v_L$ which only depends on the geometry of $K$ and $L$, e.g.:

$$\mu(v, w; K, L) = \frac{\text{vol}(K)v + \text{vol}(L)w}{\text{vol}(K) + \text{vol}(L)} \quad (67)$$

**Proposition 3** Combining (61), (62), (64), (65), (66) and (67), the finite volume approximation of (58) is the following system of o.d.e.'s:

$$\frac{dv_K}{dt} + \frac{1}{\text{vol}(K)} \sum_{L \in \mathcal{N}(K)} \text{area}(K \cap L) \Phi(v_K, v_L; K, L) = 0. \quad (68)$$

1 Volumes Finis Flux Caractristiques ([4])
References


